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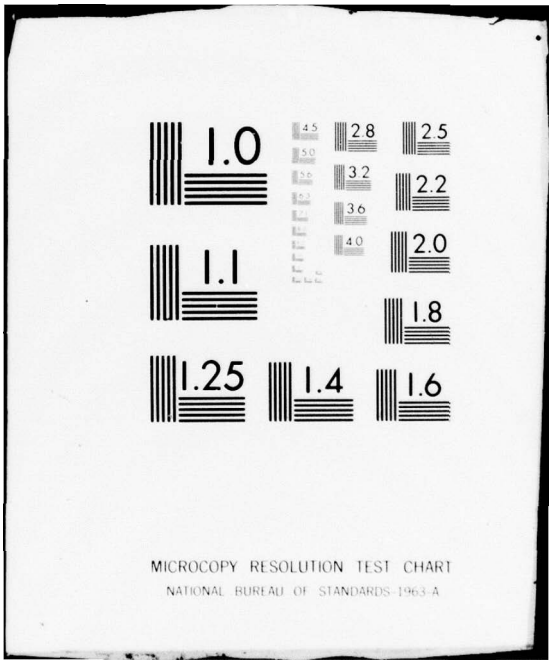
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TWO-TIME-SCALE DISCRETE SYSTEMS

RANDOLPH GALE PHILLIPS

REPORT R-839 UILU-ENG 78-2232

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The approach taken in this thesis is to decouple a two-time-scale discrete system into slow and fast subsystems. The phenomena of a boundary layer, possible order reduction, and independent slow and fast subsystems are analyzed. The analogy with continuous singularly perturbed systems is discussed throughout the text. A real time algorithm is presented which enables separate design and update of slow and fast control laws.

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TWO-TIME-SCALE DISCRETE SYSTEMS

by

RANDOLPH GALE PHILLIPS

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TWO-TIME-SCALE DISCRETE SYSTEMS

BY

RANDOLPH GALE PHILLIPS

B.E.E., Villanova University, 1976

THESIS

Submitted in partial fulfillment of the requirements
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PREFACE

This thesis serves primarily as an introduction to the analysis and design of linear shift-invariant discrete-time systems possessing a two-time-scale property. As done for continuous singularly perturbed (two-time-scale) systems, we hope to exploit the relative "spread" between eigenvalues to obtain reduced-order simulation and design. Then, we can propose computationally efficient algorithms to generate the control law to be implemented on the high order model.

In Section 1.2 of the next chapter we give basic definitions and requirements for discrete-time systems to possess the two-time scale property.

In Chapter 2, we propose a general class of discrete-time systems that will satisfy the two-time-scale requirement if a parameter μ is sufficiently small. Then, we outline how a general discrete system may be "fitted" to our model. By slightly modifying the derivations in [6], we derive sufficiency conditions on our μ parameter such that our two limiting cases, denoted "H" and "V" satisfy the two-time-scale property. This will enable a "slow-fast" block diagonalization to be possible. Finally, we conclude this chapter with sufficiency conditions for "slow-fast" block diagonalization of our general form and with some comments on the dual nature of our transformations.

Chapter 3 uses the recursive property of discrete time systems to show how truncated versions of the N-delayed sequence of $x_1(k+1)$ may be used to approximate the systems fast modes as delays. Also, by assuming that A_{11}^{-1} exists both forward and reverse recursions are possible. This enables

us to "reflect ahead" terms in the series to determine the homogeneous structure at some desired future iteration. This idea is applied to both homogeneous and nonhomogeneous systems.

In Chapter 4, $0(\mu)$ and $0(\mu^2)$ approximations of control laws are designed for the slow modes of near limiting cases. Then, high-accuracy reduced-order models for eigenvalue placement are proposed for each limiting case. This is then extended to our general model and an efficient algorithm is proposed for use in a multiprocessing control environment.

TABLE OF CONTENTS

1. INTRODUCTION - BASIC CONCEPTS

1.1. Problem Statement

Methods for approximate control of large scale systems have received a great deal of attention in recent works. Of these methods, aggregation, disaggregation, and singular perturbations seem to be the most well known [1]. The analysis and design of continuous linear singularly perturbed (two-time-scale) systems has been well documented [2,3,4]. The multiple-time-scale property of these systems has been used in arriving at reduced order models and control laws for these higher order "stiff" models.

Until recently, just about all the work done on systems possessing a multiple-time-scale property has been done on continuous systems. The area of discrete two-time-scale systems has received little attention, and with the increase in computer-based adaptive control schemes in a time-sharing environment, the need for a thorough discrete analysis and a simplified design procedure for systems possessing this property becomes essential.

The basic model assumed for the system dynamics is a set of N linear shift-invariant difference equations of the form

$$x(k+1) = Ax(k) + Bu(k) \tag{1.1}$$

where x is the vector state of the system belonging to an N-dimensional vector space, u is the vector input belonging to an M-dimensional vector space, and the matrices are assumed to have constant coefficients and be of compatible size with the above vectors. The variable k is integer valued.

There are three important sources of discrete-time models [9]. One is for the approximation, usually for the purpose of digital simulation, of the system of continuous differential equations

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (t_o \leq t \leq t_f) \tag{1.2}$$

by corresponding difference equations. For example, if a first difference is used to approximated a derivative then the following approximation results.

$$x(t_i + \Delta) = [I + A\Delta]x(t_i) + B\Delta u(t_i) \tag{1.3}$$

where Δ is the time-discretation interval. For a discrete model obtained in this way, $(I + A\Delta)$ is generally nonsingular since Δ must be small for a good approximation to a continuous time system.

The application of Euler's approximation to continuous singularly perturbed systems has been investigated in [10] and has produced interesting results.

Another source of the discrete-time model is sampled-data systems. In sampled-data systems, a continuous system, such as represented by (1.2), is driven by an input specified at discrete-time points and has output and state variable analytically available only at discrete-time points.

The standard example of a sampled-data system follows when u(t) is a piecewise constant function of time, i.e.

$$u(t) = u(t_i) \qquad t_i \leq t < t_{i+1}$$

and the state and output are sampled at discrete time points t_i. The sampled-data discrete-time model can then be computed from the continuous

differential equations by using $u(t) = u(t_i)$, then

$$x(t) = \Phi(t,t_i)x_i + \int_{t_i}^{t} \Phi(t,\tau)Bd\tau\, u(t_i) \quad (t_i \le t \le t_f). \tag{1.4}$$

By letting $t = t_{i+1}$ we obtain our discrete-time model with

$$A = \Phi(t_{i+1},t_i), \qquad B = \int_{t_i}^{t_{i+1}} \Phi(t_{i+1},\tau)Bd\tau.$$

Since the transition matrix $\Phi(t,\tau)$ is nonsingular, the matrix A is nonsingular. This class of discrete models is ideal for computer control analysis of continuous systems.

The final class of discrete-time systems results when the physical system is inherently discrete. Such models are common in economic, biological, and sociological systems. The system matrix A for this type of system may well be singular. So, the problems associated with large scale systems are for the most part, preserved for these discrete-time models.

1.2. Basic Definitions

In this section we define what is meant by a two-time-scale linear shift-invariant discrete systems. These defining properties were originally proposed in [8].

Consider the linear shift-invariant discrete-time system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \tag{1.6}$$

where i. $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in R^N \quad u(k) \in R^m$

ii. A_{ij}, B_j $i,j = 1,2$ are real constant coefficient matrices of suitable size

iii. The eigenvalues are concentrated in two distinct groups of N_s and N_f such that

a. $N = N_s + N_f$

b. Denote $\lambda_f \triangleq \max_i |\lambda_i| \; i \in N_f$

$\lambda_s \triangleq \min_j |\lambda_j| \; j \in N_s$

then

$$\lambda_f \ll \lambda_s \ll 0(1/\lambda_f)$$

or, the maximum possible magnitude of any eigenvalue of our system must be within some neighborhood of 1.

Note: By defining an upper bound on the magnitude of our eigenvalues of $0(1/\lambda_f)$, our fast modes will always be stable and only the slow modes can be unstable. This property will be assumed throughout the text.

Now, given (1.6) with conditions i, ii, and iii, then there will exist a basis in R^N such that (1.6) takes the form

$$\begin{bmatrix} x_s(k+1) \\ \hline x_f(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_s & 0 \\ \hline 0 & A_f \end{array}\right] \begin{bmatrix} x_s(k) \\ \hline x_f(k) \end{bmatrix} + \begin{bmatrix} B_s \\ \hline B_f \end{bmatrix} u(k) \qquad (1.7)$$

where i. $x_s(k) \in R^{N_s}$

ii. $x_f(k) \in R^{N_f}$.

This transformation is not necessarily modal. However, multiple and complex conjugate eigenvalues are naturally grouped together in either A_s or A_f.

Under this transformation (1.6) is said to possess the two-time-scale property if

$$\lambda_f \triangleq \max_j |\lambda_j(A_f)|$$

$$\lambda_s \triangleq \min_i |\lambda_i(A_s)| .$$

Then

$$\lambda_f \ll \lambda_s \ll 0(1/\lambda_f).$$

In other words, if there is sufficient "gap" between the eigenvalues of A_s and A_f, then (1.6) possesses the two-time-scale property.

We can express this desired property in the form of matrix norms, since for any nonsingular matrix S,

$$\max_i |\lambda_i(S)| \leq \|S\|$$

$$[\min_j |\lambda_j(S)|]^{-1} \leq \|S^{-1}\|$$

where we will define our norm as

$$\|A\| = [\lambda_{max}[A^*A]]^{\frac{1}{2}} \quad * \triangleq \text{conjugate transpose.}$$

Therefore,

$$\min_i |\lambda_i(A_s)| \geq \|A_s^{-1}\|^{-1} \quad \text{(lower bound)}$$

$$\max_j |\lambda_j(A_f)| \leq \|A_f\| \quad \text{(upper bound)}$$

and our requirement for a two-time-scale property in terms of these bounds becomes

$$\|A_s^{-1}\|^{-1} \gg \|A_f\| .$$

Next, we will investigate a general system structure for which there exists a basis such that decomposition into our block diagonal form is assured providing a certain parameter is sufficiently small.

2. A CLASS OF TWO-TIME-SCALE DISCRETE SYSTEMS

2.1. Introduction

In this chapter, we introduce a model for a class of two-time-scale discrete systems. We then give a simple algorithm that will attempt to fit any discrete-time system to our general model.

In the remaining sections of this chapter we attempt to isolate the slow and fast subsystems through similarity transformations. The original proof of existence of these transformations was used in the study of continuous singularly perturbed system. However, we will show that by modifying the derivation, we can adapt these ideas to a class of discrete systems. Also, we can define two unique transformations, one giving computational simplicity to the slow and one to the fast subsystems. These transformations will become very important in our design chapter.

We conclude this chapter with a brief discussion on degenerate systems and some comments on the dual nature of the transformations.

2.2. System Forms

A general class of linear shift-invariant discrete-time systems possessing a two-time-scale property may be expressed as

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & \mu^{(1-j)}\hat{A}_{12} \\ \hline \mu^{j}\hat{A}_{21} & \mu\hat{A}_{22} \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u(k) \qquad (2.1)$$

where

i. $x_1(k) \in R^{N_s}$, $x_2(k) \in R^{N_f}$, $u(k) \in R^m$.

ii. $\hat{A}_{\ell m}$, $\hat{B}_m$, $\ell, m = 1,2$ are real constant coefficient matrices of appropriate order.

iii. $0 \leq j \leq 1$.

iv. μ is a small positive parameter such that $0 < \mu < 1$.

Note: Throughout the text, the following notation will be consistent

$$A_{ij} = \mu^x \hat{A}_{ij}, \quad B_j = \mu^x \hat{B}_j \qquad x = 1, 1-j, j.$$

For $j = 1$ and $j = 0$ we define two limiting cases which we will show to have dual characteristics to one another. These limiting structures are

Structure "H" $\triangleq$ $(j=1)$

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline \mu A_{21} & \mu A_{22} \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u(k). \tag{2.2}$$

Structure "V" $\triangleq$ $(j=0)$

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & \mu\hat{A}_{12} \\ \hline A_{21} & \mu\hat{A}_{22} \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u(k). \tag{2.3}$$

For $0 < j < 1$ we will prove that the general form satisfies the two-time-scale property under similar sufficiency conditions as either of the limiting cases. In this analysis, however, when $0(\mu)$ and $0(\mu^2)$ reduced order models are considered, only limiting or near limiting cases will be of practical importance.

In the next section, the problem of putting a general two-time-scale discrete system into our general form is considered.

2.3. Permutation of States and Establishing of μ Parameter

In this section, we outline a straightforward procedure for transforming an Nth order discrete-time system into our proposed general form. This procedure could be easily programmed for easy analysis of any discrete system.

Given the linear shift invariant discrete-time system

$$x(k+1) = Ax(k) + Bu(k) \tag{2.4}$$

where i. $\underline{x}(k) \in R^N$, $\underline{u}(k) \in R^M$

ii. A,B are real constant coefficient matrices.

iii. There exist two distinct groups of open loop eigenvalues numbering N_s and N_f such that

a. $N_s + N_f = N$

b. If $\lambda_f \triangleq \max_j |\lambda_j| \in N_f$

$\lambda_s \triangleq \min_i |\lambda_i| \; i \in N_s$

then $\lambda_f << \lambda_s << 0(1/\lambda_f)$.

Permute the states finding the ratio of the norm of the homogeneous part of N_f of the states to the norm of the homogeneous part of the remaining N_s states. Continue this until all state combinations have been tried. Retain the permuted order such that

$$\frac{||D||}{||A||} \triangleq \text{minimum}$$

and the permuted system looks like

$$\begin{bmatrix} y(k+1) \\ \hline z(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \begin{bmatrix} y(k) \\ \hline z(k) \end{bmatrix} \tag{2.5}$$

where $y(k) \in R^{N_s}, \quad z(k) \in R^{N_f}$.

Once this minimum has been found, we have identified our μ parameter as

$$\mu = \frac{\|D\|}{\|A\|}.$$

By evaluating $\|B\|$ and $\|C\|$, we can then establish whether or not our general model can fit the system. We now discuss some different cases when evaluating these norms.

i. If $\|B\|$, $\|C\|$ are both on the order of $\|A\|$ or greater, then the system cannot be put into our general form.

ii. For the remaining cases, a closer examination must be made. First, define the constants

$$\alpha = \frac{\|C\|}{\|A\|}, \qquad \gamma = \frac{\|B\|}{\|A\|}.$$

Then find $\quad (1-j) \triangleq \theta = \dfrac{\log(\gamma)}{\log(\mu)} > 0$

$$j \triangleq \beta = \frac{\log(\alpha)}{\log(\mu)} > 0.$$

We now can give a general system form which has our class of systems as a special case

$$\begin{bmatrix} y(k+1) \\ \hline z(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A & \mu^{\theta} B^* \\ \hline \mu^{\beta} C^* & \mu D^* \end{array}\right] \begin{bmatrix} y(k) \\ \hline z(k) \end{bmatrix}$$

where $\quad \mu^F B^* = B, \quad \mu^B C^* = C, \quad \mu D^* = D.$

Now, by observing the sum $\beta+\theta$, we can cover all possible cases

Case a: $\quad 1-\varepsilon \leq \beta+\theta \leq 1+\varepsilon \qquad 0 < \varepsilon \ll 1$

or, for $\beta+\theta$ within some small neighborhood of 1.

If $\beta+\theta$ satisfies these bounds, the system is referred to as "normal," and has the following properties:

1) The system fits our model with $\theta \triangleq (1-j)$ and $\beta \triangleq (j)$.

2) The new subsystem matrices B^*, C^*, D^* have all been normalized with respect to A and therefore,

$$\|A\| \approx \|B^*\| \approx \|C^*\| \approx \|D^*\|.$$

Case b: $0 \leq \theta+\beta < 1-\varepsilon$

With these bounds, we are approaching condition i. Therefore, the system usually cannot fit the model.

Case c: $1+\varepsilon < \beta+\theta < \infty$

In this case, the system is referred to as "degenerate," and has the following properties:

1) The system can be made to fit our model by appropriately scaling down x and/or y such that

$$\beta^* = \beta - \delta\beta$$

$$\theta^* = \theta - \delta\theta$$

until $\theta^*+\beta^*$ satisfies case a, then, the norm of B^* will be scaled down by $\mu^{\delta\theta}$ and C^* will be scaled down by $\mu^{\delta\beta}$ resulting in

$$\|A\| \approx \|D^*\| > \|B^*\|, \quad \|C^*\|.$$

2) The system has "weaker" coupling between subsystems than a "normal" system with as similar value of μ.

As shown in a later section, a degenerate system will have some advantages over its corresponding normal form.

Thus, if $\theta+\beta$ satisfy the following bounds

$$1-\varepsilon < \theta+\beta < \infty \qquad 0 < \varepsilon \ll 1.$$

the corresponding system can be fit to our model. Note that for $\theta \approx 1$, $\beta \approx 0$ or $\theta \approx 0$, $\beta \approx 1$ we can identify one of our limiting cases. Approximating near-limiting cases by limiting cases becomes important when reduced-order models are considered. We now give a simple example.

Example: Given the system

$$\begin{bmatrix} y_1(k+1) \\ y_2(k+1) \\ y_3(k+1) \\ y_4(k+1) \end{bmatrix} = \begin{bmatrix} 1.3 & .7 & -.2 & 0 \\ .1 & .1 & -.1 & 0 \\ .6 & .8 & 1.0 & .4 \\ .1 & .1 & 0 & .05 \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \\ y_3(k) \\ y_4(k) \end{bmatrix}$$

eigenvalues are $1.16 \pm .37j,\ .1,\ .02.$

Permute the states by

$$x(k) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} y(k).$$

Then

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 1.0 & .6 & .4 & .8 \\ -.2 & 1.3 & 0 & .7 \\ 0 & .1 & .05 & .1 \\ -.1 & .1 & 0 & . \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}.$$

Identify

$$A = \begin{bmatrix} 1.0 & .6 \\ -.2 & 1.3 \end{bmatrix} \qquad B = \begin{bmatrix} .4 & .8 \\ 0 & .7 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & .1 \\ -.1 & .1 \end{bmatrix} \qquad D = \begin{bmatrix} .05 & .1 \\ 0 & .1 \end{bmatrix}$$

$$\|A\| = 1.46, \quad \|B\| = 1.11, \quad \|C\| = .173, \quad \|D\| = .1414$$

$$\frac{\|D\|}{\|A\|} \triangleq \mu = .09685$$

$$\alpha \triangleq \frac{\|C\|}{\|A\|} = .11849 \qquad \gamma \triangleq \frac{\|B\|}{\|A\|} = .7603$$

$$j \triangleq \beta = \frac{\log[\alpha]}{\log[\mu]} = .91362$$

$$(1-j) \triangleq \theta = \frac{\log[\gamma]}{\log[\mu]} = .11738$$

$$\theta+\beta = 1.031 \approx 1 \text{ (normal)}.$$

Therefore the system fits our model and in this form looks like

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \left[\begin{array}{cc|cc} 1.0 & .6 & \mu^{(.11738)} \begin{pmatrix} .5261 & 1.052 \\ 0 & .921 \end{pmatrix} \\ -.2 & 1.3 & \\ \hline \mu^{(.91362)} \begin{pmatrix} 0 & .844 \\ -.844 & .844 \end{pmatrix} & & \mu \begin{pmatrix} .5162 & 1.0325 \\ 0 & 1.0325 \end{pmatrix} \end{array}\right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}.$$

But notice, $\mu^{(.11738)}(B^*)$ is close to B. Therefore, to approximate this as a limiting case, ideal for perturbations analysis, let

$$\mu^{.11738} \approx 1, \quad \mu^{.91362} \approx \mu$$

which identifies limiting case H and looks like

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 1 & .6 & .4 & .8 \\ -.2 & 1.3 & 0 & .7 \\ \mu \begin{pmatrix} 0 & 1.032 \\ -1.032 & 1.032 \end{pmatrix} & & \mu \begin{pmatrix} .5162, & 1.0352 \\ 0 & , 1.0352 \end{pmatrix} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

$$\mu = .09685.$$

In the next few sections the value of μ becomes critical in determining whether or not a system can be exactly decomposed into slow-fast subsystems. If conditions are not met, a scaling of states as well as a permutation may be necessary.

2.4. Slow-Fast Block Diagonalization Using Riccati Type Equations

For continuous systems possessing a two-time-scale property, [6] has developed sufficient conditions to block diagonalize the system so that one subsystem contains <u>all</u> N_s slow modes and the other subsystem contains <u>all</u> N_f fast modes. The sufficient condition is based on the norms of the original subsystem matrices. By slightly modifying the derivation, we can apply Lemma 1 of [6] to achieve sufficiency conditions needed to block diagonalize both structure "H" and "V" such that they take the form

$$\begin{bmatrix} x_s(k+1) \\ x_f(k+1) \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_f \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} u(k) \tag{2.6}$$

and the submatrices A_s and A_f satisfy the two-time-scale property.

Our purpose for handling only these two limiting structures is so that we can first understand the origin of our dual transformations. These two system structures motivate straightforward methods of solving dual algebraic Riccati equations needed to define our slow-fast transformations. These solutions are found using a successive approximations scheme where it is critical what we select as an initial iterate. These ideas will carry over to the next section where these transformations become special cases of our general class ($0 \leq j \leq 1$).

Part 1: Block diagonalization of form "H"

Given

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline \mu\hat{A}_{21} & \mu\hat{A}_{22} \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix}$$

let

$$y_1(k) = x_1(k) + Lx_2(k) \tag{2.8}$$

$$\begin{aligned} y_1(k+1) &= x_1(k+1) + Lx_2(k+1) \\ &= A_{11}x_1(k) + A_{12}x_2(k) + \mu L\hat{A}_{21}x_1(k) + \mu L\hat{A}_{22}x_2(k) \\ &= (A_{11}+\mu L\hat{A}_{21})y_1(k) + (-A_{11}L+A_{12}+\mu L\hat{A}_{22}-\mu L\hat{A}_{21}L)x_2(k). \end{aligned}$$

We want to find L such that

$$-A_{11}L + A_{12} + \mu L\hat{A}_{22} - \mu L\hat{A}_{21}L = 0. \tag{2.9}$$

In [11] a unique solution to this equation is found through partitioned submatrices of the modal matrix. If only an approximation of L, namely L_o, is available, an efficient method of improving this initial guess can be obtained using a successive approximation scheme on the algebraic Riccati equation (2.9) with L_o as the first iterate.

In our case, the calculation of a modal matrix is something we computationally want to avoid. In general, there still remains the problem of picking a good first guess L_o. For this limiting structure, however, the choice is obvious. For μ small,

$$\begin{aligned} -A_{11}L + A_{12} + 0(\mu) &= 0 \\ L_o &= A_{11}^{-1}A_{12}. \end{aligned}$$

If A_{11}^{-1} exists, which we will show to be a necessary condition for (2.1) to possess a two-time-scale property, we can define the following successive approximation scheme.

Letting $L = L_o + D$ and substituting into (2.9) we get

$$-A_{11}(L_o+D) + A_{12} + \mu(L_o+D)\hat{A}_{22} - \mu(L_o+D)\hat{A}_{21}(L_o+D) = 0$$

$$-A_{11}D + \mu L_o A_o + \mu D A_o - \mu L_o \hat{A}_{21} D - \mu D \hat{A}_{21} D = 0 \qquad (2.10)$$

where

$$A_o = \hat{A}_{22} - \hat{A}_{21} L_o$$

and, since A_{11}^{-1} exist, we naturally define the successive approximation scheme

$$D_{k+1} = \mu A_{11}^{-1}[L_o \hat{A}_{21} D_k + D_k \hat{A}_{21} D_k - L_o A_o - D_k A_o]. \qquad (2.11)$$

If we define

$$a = \mu\|A_o\|, \quad b = \|\hat{A}_{21}\| \, \|L_o\|, \quad c = \|A_{11}^{-1}\|$$

$$d_k = \frac{\|D_k\|}{\mu\|A_o\|\|L_o\|}$$

we can apply the contraction mapping argument of [6].

Lemma 1: If

$$c < \frac{1}{3}(a+b)^{-1}$$

or if

$$\|A_{11}^{-1}\| < \frac{1}{3}(\mu\|A_o\| + \mu\|\hat{A}_{21}\| \, \|L_o\|)^{-1}. \qquad (2.12)$$

There exists a unique real root of (2.10) bounded by

$$0 \le \|d_f\| \le 2(a+b)^{-1}$$

$$0 \le \|D\| \le \frac{2\|A_o\| \, \|L_o\|}{\|A_o\| + \|\hat{A}_{21}\| \, \|L_o\|}.$$

Moreover, this root is an asymptotically stable equilibrium of (2.11).

Proof: Appendix A.

Due to the explicit presence of μ, we state an obvious sufficient bound on μ for convergence of (2.11).

Corollary 2: To put structure H in lower block triangular form, it is sufficient that μ satisfy the following bounds

$$0 \leq \mu < \frac{1}{3\|A_{11}^{-1}\|\,(\|A_o\| + \|\hat{A}_{21}\|\;\|L_o\|)}. \tag{2.13}$$

The proof follows directly from equation (2.12) of the previous lemma.

To complete the block diagonalization, let

$$y_2(k) = x_2(k) - My_1(k) \tag{2.14}$$

$$\begin{aligned} y_2(k+1) &= x_2(k+1) - My_1(k+1) \\ &= \mu\hat{A}_{21}y_1(k) + \mu(\hat{A}_{22}-\hat{A}_{21}L)x_2(k) - M(A_{11}+\mu L\hat{A}_{21})y_1(k) \\ &= \mu(\hat{A}_{22}-\hat{A}_{21}L)y_2(k) + (\mu(\hat{A}_{22}-\hat{A}_{21}L)M - M(A_{11}+\mu L\hat{A}_{21}) + \mu\hat{A}_{21})y_1(k). \end{aligned}$$

We seek M such that

$$\mu(\hat{A}_{22}-\hat{A}_{21}L)M - M(A_{11}+\mu L\hat{A}_{21}) + \mu\hat{A}_{21} = 0. \tag{2.15}$$

By identifying

$$A_s = \mu(\hat{A}_{22}-\hat{A}_{21}L)$$

$$A_s = A_{11} + \mu L\hat{A}_{21}$$

(2.15) becomes

$$A_f M - MA_s + \mu\hat{A}_{21} = 0. \tag{2.16}$$

From [14] we know that this Lyapunov equation will have a unique solution since A_s and A_f have no eigenvalues in common.

Again, through successive approximation, we can show that M is an asymptotically stable equilibrium of the difference equation

$$M_{k+1} = \mu[\hat{A}_{22}M_k - \hat{A}_{21}LM_k - M_kL\hat{A}_{21} + \hat{A}_{21}]A_{11}^{-1}. \tag{2.17}$$

Lemma 2 of [6] again use a contraction mapping argument to provide a sufficient condition for convergence of (2.17).

Lemma 3: If $$c < \frac{1}{2}(a+b)^{-1}$$

or

$$\|A_{11}^{-1}\| < \frac{1}{2\mu}\left(\|A_o\| + \|\hat{A}_{21}\|\ \|L_o\|\right)^{-1}$$

then (2.16) has a unique real root such that $\|M\| \triangleq 0(\mu)$. Moreover, this root is an asymptotically stable equilibrium of (2.17)

Proof: Appendix A.

Our transformed system now becomes

$$\begin{bmatrix} y_1(k+1) \\ \hline y_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} (A_{11}+\mu L\hat{A}_{21}) & 0 \\ \hline 0 & \mu(\hat{A}_{22}-\mu\hat{A}_{21}L) \end{array}\right] \begin{bmatrix} y_1(k) \\ \hline y_2(k) \end{bmatrix}.$$

To verify the two-time-scale nature of this system, we know

$$\|A_s\| = \|A_{11}+\mu L\hat{A}_{21}\|$$
$$\|A_f\| = \mu\|\hat{A}_{22}-\hat{A}_{21}L\|.$$

In Chapter 1 we showed that for a linear shift-invariant discrete-time system to possess a two-time-scale property it was sufficient that

$$\|A_s^{-1}\|^{-1} >> \|A_f\|.$$

In our case we need

$$\|(A_{11}+\mu L\hat{A}_{21})^{-1}\|^{-1} >> \mu\|\hat{A}_{22}-\hat{A}_{21}L\|$$

which will be true for every μ satisfying Corollary 2. Putting this transformation into matrix form we have

$$\begin{bmatrix} y_1(k) \\ \hline y_2(k) \end{bmatrix} = \left[\begin{array}{c|c} I_1 & \bar{L} \\ \hline -M & I_2-ML \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} \tag{2.18}$$

where $y_1(k) \in R^{N_s}$, $y_2(k) \in R^{N_f}$, and I_1 and I_2 are N_s and N_f order identity matrices respectively. This nonsingular transformation has the attractive

feature that its inverse is simply given as

$$\begin{bmatrix} x_1(k) \\ - - - \\ x_2(k) \end{bmatrix} = \begin{bmatrix} I_1 - LM & \vdots & -L \\ - - - & \vdots & - - \\ M & \vdots & I_2 \end{bmatrix} \begin{bmatrix} y_1(k) \\ - - - \\ y_2(k) \end{bmatrix}.$$

Applying this transformation to our nonhomogeneous system "H"

$$\begin{bmatrix} y_1(k+1) \\ - - - - - \\ y_2(k+1) \end{bmatrix} = \begin{bmatrix} (A_{11}+\mu L\hat{A}_{21}) & \vdots & 0 \\ - - - - & \vdots & - - - - \\ 0 & \vdots & \mu(\hat{A}_{22}-\hat{A}_{21}L) \end{bmatrix} \begin{bmatrix} y_1(k) \\ - - - - \\ y_2(k) \end{bmatrix} + \begin{bmatrix} B_1+LB_2 \\ - - - - - - \\ -MB_1+(I-ML)B_2 \end{bmatrix} u(k)$$

with

$$y_1(0) = x_1(0) + Lx_2(0) \tag{2.19}$$

$$y_2(0) = -Mx_1(0) + (I-ML)x_2(0).$$

Note that the $y_1(k+1)$ subsystem is totally independent of the M transformation matrix.

Part 2: Block diagonalization of form "V"

Given

$$\begin{bmatrix} x_1(k+1) \\ - - - \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & \vdots & \mu\hat{A}_{12} \\ - - & \vdots & - - \\ A_{21} & \vdots & \mu\hat{A}_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ - - \\ x_2(k) \end{bmatrix}$$

let

$$y_2(k) = x_2(k) + Lx_1(k) \tag{2.20}$$

$$\begin{aligned} y_2(k+1) &= x_2(k+1) + Lx_1(k+1) \\ &= A_{21}x_1(k) + \mu\hat{A}_{22}x_2(k) + LA_{11}x_1(k) + \mu L\hat{A}_{12}x_2(k) \\ &= (\mu\hat{A}_{22}+\mu L\hat{A}_{12})y_2(k) + (A_{21}-\mu\hat{A}_{22}L+LA_{11}-\mu L\hat{A}_{12}L)x_1(k). \end{aligned}$$

We want L such that

$$A_{21} - \mu\hat{A}_{22}L + LA_{11} - \mu L\hat{A}_{12}L = 0. \tag{2.21}$$

For μ small a good first approximation would be

$$A_{21} + LA_{11} + 0(\mu) = 0$$

$$L_o = -A_{21}A_{11}^{-1}.$$

We can again apply the contraction mapping argument of [6]. Letting

$$L = L_o + D$$

$$A_{21} - \mu\hat{A}_{22}(L_o+D) + (L_o+D)A_{11} - \mu(L_o+D)\hat{A}_{12}(L_o+D) = 0$$

$$-\mu A_o L_o - \mu A_o D + DA_{11} - \mu D\hat{A}_{12}L_o - \mu D\hat{A}_{12}D = 0 \qquad (2.23)$$

where

$$A_o = \hat{A}_{22} + L_o\hat{A}_{12}.$$

Since A_{11}^{-1} exists, we again naturally apply the successive approximation scheme

$$D_{k+1} = \mu(A_oL_o + A_oD_k + D_k\hat{A}_{12}L_o + D_k\hat{A}_{12}D_k)A_{11}^{-1}.$$

If we define

$$a = \mu\|A_o\| \qquad b = \mu\|\hat{A}_{12}\|\ \|L_o\| \qquad c = \|A_{11}^{-1}\|$$

$$d_k = \frac{\|D_k\|}{\mu\|A_o\|\ \|L_o\|}.$$

We can again apply Lemma 1 of [6].

Lemma 4: If

$$c < \tfrac{1}{3}(a+b)^{-1}$$

or

$$\|A_{11}^{-1}\| < \frac{1}{3\mu}(\|A_o\| + \|\hat{A}_{12}\|\ \|L_o\|)^{-1}. \qquad (2.24)$$

Then there exists a unique real root of (2.22) bounded by

$$0 \le \|D\| \le \frac{2\|A_o\|\ \|L_o\|}{\|A_o\| + \|\hat{A}_{12}\|\ \|L_o\|}$$

such that D is an asymptotically stable equilibrium of (2.23).

Proof: Appendix A.

We now give the obvious bounds on μ.

Corollary 5: To transform form "V" into upper block diagonal form, it is sufficient that μ be within the bounds defined by

$$0 \leq \mu < \frac{1}{3\|A_{11}^{-1}\| (\|A_o\| + \|\hat{A}_{12}\| \ \|L_o\|)} . \tag{2.25}$$

The proof follows directly from (2.24) of Lemma 5. To complete the block diagonalization let

$$y_1(k) = x_1(k) - My_2(k)$$

$$y_1(k+1) = x_1(k+1) - My_2(k+1)$$

$$= [A_{11} - \mu\hat{A}_{21}L]y_1(k) + [(A_{11} - \mu\hat{A}_{21}L)M - M(\mu\hat{A}_{22} + \mu L\hat{A}_{12}) + \mu\hat{A}_{12}]y_2(k).$$

We can identify

$$A_s = A_{11} - \mu\hat{A}_{21}L$$

$$A_f = \mu(\hat{A}_{22} + L\hat{A}_{12}).$$

Therefore, we want M such that

$$A_sM - MA_f + \mu\hat{A}_{12} = 0. \tag{2.26}$$

Again from [14] we know that this Lyapunov equation will have a unique solution since A_s and A_f have no eigenvalues in common. Through successive approximation, we can show that M is an asymptotically stable equilibrium of

$$M_{k+1} = \mu A_{11}^{-1}(\hat{A}_{12} - LM_k + M_k\hat{A}_{22} + M_kL\hat{A}_{12} - \hat{A}_{12}). \tag{2.27}$$

Lemma 6: If

$$c < \frac{1}{2} (a+b)^{-1}$$

or

$$\|A_{11}^{-1}\| < \frac{1}{2\mu} (\|A_o\| + \|\hat{A}_{12}\| \ \|L_o\|)^{-1}.$$

Then the root to (2.26) is an asymptotically stable equilibrium of (2.27).

Proof: Appendix A.

Our transformed system now looks like

$$\begin{bmatrix} y_1(k+1) \\ \hline y_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11}-\mu\hat{A}_{12}L & \vdots & 0 \\ \hline 0 & \vdots & \mu(\hat{A}_{22}+L\hat{A}_{12}) \end{bmatrix} \begin{bmatrix} y_1(k) \\ \hline y_2(k) \end{bmatrix}$$

and again it is obvious that our two time scale property is satisfied since

$$\|(A_{11}-\mu\hat{A}_{12}L)^{-1}\|^{-1} >> \mu\|\hat{A}_{22}+L\hat{A}_{12}\|.$$

This transformation in matrix form becomes

$$\begin{bmatrix} y_1(k) \\ \hline y_2(k) \end{bmatrix} = \begin{bmatrix} I_1-ML & \vdots & -M \\ \hline L & \vdots & I_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} \tag{2.28}$$

where $y_1(k) \in R^{N_s}$, $y_2(k) \in R^{N_f}$ and I_1, I_2 are N_s and N_f order identity matrices respectively. This also has a simple inverse given by

$$\begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} = \begin{bmatrix} I_1 & \vdots & M \\ \hline -L & \vdots & I_2-LM \end{bmatrix} \begin{bmatrix} y_1(k) \\ \hline y_2(k) \end{bmatrix}.$$

Applying this transformation to our original nonhomogeneous system we get

$$\begin{bmatrix} y_1(k+1) \\ \hline y_2(k+1) \end{bmatrix} = \begin{bmatrix} (A_{11}-\mu\hat{A}_{12}L) & \vdots & 0 \\ \hline 0 & \vdots & \mu(\hat{A}_{22}+L\hat{A}_{12}) \end{bmatrix} \begin{bmatrix} y_1(k) \\ \hline y_2(k) \end{bmatrix} + \begin{bmatrix} (I-ML)B_1-MB_2 \\ \hline LB_1+B_2 \end{bmatrix} u(k) \tag{2.29}$$

where

$$y_1(0) = (I-ML)x_1(0) - Mx_2(0)$$

$$y_2(0) = Lx_1(0) + x_2(0).$$

Note, in this case the $y_2(k+1)$ subsystem is independent of the M transformation matrix. This is dual to the results obtained in part one, and will become very important later on.

2.5. General Case - Block Transformation

Up to now, we have dealt exclusively with our limiting case models "H" and "V." This was done to introduce the reader to the two types of transformations and to easily see the formulation of the matrix Riccati recursion equations as an application of [6] and an analog to what was done in [5]. What we do now is to apply <u>both</u> transformations to the general model and not to isolate limiting cases. This will enable us to develop sufficiency conditions for either transformation to be applicable to the general model. This idea will become extremely important when reduced order control laws are considered.

Since many of the derivations are similar to section (2.4), some of the repititious steps will be eliminated. Given

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & \mu^{(1-j)}\hat{A}_{12} \\ \mu^{j}\hat{A}_{21} & \mu A_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{2.30}$$

where

i. All matricies have constant coefficients

ii. $x_1(k) \in R^{N_s}$, $x_2(k) \in R^{N_f}$

iii. $0 \leq j \leq 1$

iv. μ is some small positive parameter < 1.

Representing $\mu^{x}\hat{A}_{ij}$ by A_{ij}, let us consider the general transformation

$$y_2(k) = x_2(k) + Px_1(k) \quad \text{where } P \Rightarrow L \text{ as } j \Rightarrow 1.$$

Our notation has been changed due to our convention in the limiting cases that $L \triangleq 0(1)$ and $M \triangleq 0(\mu)$.

Even in this general case, P assumes the role of the independent transformation matrix. And, as a result, the homogeneous submatrix <u>and</u>

control submatrix of the $y_2(k)$ (fast) state variables will be dependent only on the P transformation matrix. We will denote this as the "F" transformation. Continuing the derivation

$$y_2(k+1) = x_2(k+1) + Px_1(k+1) \tag{2.31}$$
$$= [A_{21} + PA_{11} - A_{22}P - PA_{12}P]x_1(k) + [A_{22} + PA_{12}]y_2(k)$$

and obviously we want

$$A_{21} + PA_{11} - A_{22}P - PA_{12}P = 0 \tag{2.32}$$

which is equivalent to (2.21) except for $P = L$. Again, we would like to apply the contraction mapping argument of [6] and obtain a convergent successive approximation scheme so as to be able to iterate a solution to (2.32). However, an initial guess to such an iteration is not as obvious as before.

Make the substitutions

$$A_{21} = \mu^j \hat{A}_{21}, \quad A_{22} = \mu \hat{A}_{22}, \quad A_{12} = \mu^{(1-j)} \hat{A}_{12} \tag{2.33}$$

in (2.32), and we obtain

$$\mu^j \hat{A}_{21} + PA_{11} - \mu \hat{A}_{22}P - P\mu^{(1-j)}\hat{A}_{12}P = 0.$$

For μ small and $(0 \leq j \leq 1)$, a good $0(\mu)$ initial guess is not so obvious. Let us try a general form of L_o used in (2.21)

$$P_o = -\mu^j A_{21} A_{11}^{-1}.$$

Substituting this into (2.32) we get

$$\mu^j \hat{A}_{21} + (-\mu^j \hat{A}_{21} A_{11}^{-1})A_{11} + \mu \hat{A}_{22} \mu^j \hat{A}_{21} A_{11}^{-1} - \mu^j \hat{A}_{21} A_{11}^{-1} \mu^{(1-j)} \hat{A}_{12} \hat{A}_{21} \mu^j A_{11}^{-1} = 0$$
$$\mu^{(1+j)} \hat{A}_{22} \hat{A}_{21} A_{11}^{-1} - \mu^{(1+j)} \hat{A}_{21} A_{11}^{-1} \hat{A}_{12} \hat{A}_{21} A_{11}^{-1} = 0 \tag{2.34}$$

or
$$P = P_o + O(\mu^{(1+j)})$$
$$= -\mu^j A_{21}A_{11}^{-1} + O(\mu^{1+j}).$$

Thus, $P-P_o \triangleq O(\mu^{1+j})$ and P_o will always be an order of μ greater than any error between P and P_o.

Now, since A_{11}^{-1} exists, we can naturally define the successive approximation scheme to improve our solution to P as

$$P_{k+1} = (A_{22}P_k + P_kA_{12}P_k - A_{21})A_{11}^{-1} \tag{2.35}$$

where P_o is specified to be $-A_{21}A_{11}^{-1}$. Thus, if we can show the $D = P-P_o$ is an asymptotically stable equilibrium or

$$D_{k+1} = (A_oP_o + A_oP_k + D_kA_{12}P_o + D_kA_{12}D_k)A_{11}^{-1} \tag{2.36}$$

where
$$A_o = A_{22} + P_oA_{12}.$$

We can guarantee the P is an asymptotically stable equilibrium of (2.35). If we define

$$a = \|A_o\|, \quad b = \|P_o\| \ \|A_{12}\|, \quad c = \|A_{11}^{-1}\|, \quad d_k = \frac{\|D_k\|}{\|A_o\| \ \|P_o\|}.$$

We can again apply the contraction mapping argument of [6]. Therefore, the iteration converges when

$$c < \tfrac{1}{3}\,(a+b)^{-1}$$

$$\|A_{11}^{-1}\| < \tfrac{1}{3}\,(\|A_{22} + P_oA_{12}\| + \|P_o\| \ \|A_{12}\|)^{-1}. \tag{2.37}$$

Making substitutions (2.33)

$$\|A_{11}^{-1}\| < \tfrac{1}{3}\,(\|\mu\hat{A}_{22} - \mu^j\hat{A}_{21}A_{11}^{-1}\mu^{1-j}\hat{A}_{12}\| + \|\mu^j\hat{A}_{21}A_{11}^{-1}\| \ \|\mu^{(1-j)}\hat{A}_{12}\|)^{-1}.$$

However, for every matrix H, and constant μ

$$\|\mu H\| = |\mu| \, \|H\|$$

$$\|A_{11}^{-1}\| < \frac{1}{3} \left(\mu\|\hat{A}_{22} - \hat{A}_{21}A_{11}^{-1}\hat{A}_{12}\| + \mu\|\hat{A}_{21}A_{11}^{-1}\| \, \|\hat{A}_{12}\|\right)^{-1}$$

or, μ is bounded by

$$0 \leq \mu < \frac{1}{3\|A_{11}^{-1}\| \left(\|\hat{A}_{22} - \hat{A}_{21}A_{11}^{-1}\hat{A}_{12}\| + \|\hat{A}_{21}A_{11}^{-1}\| \, \|\hat{A}_{12}\|\right)} . \tag{2.38}$$

Also, D will be bounded by

$$0 \leq \|D\| < \frac{2\mu^{(j)}\|A_o\| \, \|P_o\|}{\|A_o\| + \|P_o\| \, \|A_{12}\|} \tag{2.39}$$

and our upper triangular system looks like

$$\begin{bmatrix} x_1(k+1) \\ \hline y_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} (A_{11} - A_{12}P) & A_{12} \\ \hline 0 & (A_{22} + PA_{12}) \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline y_2(k) \end{bmatrix} . \tag{2.40}$$

To complete the block diagonalization

$$y_1(k) = x_1(k) - Qy_2(k) \tag{2.41}$$

$$y_1(k+1) = (A_{11} - A_{12}P)y_1(k) + [(A_{11} - A_{12}P)Q - Q(A_{22} + PA_{12}) + A_{21}]y_2(k).$$

We seek Q such that

$$(A_{11} - A_{12}P)Q - Q(A_{22} + PA_{12}) + A_{21} = 0. \tag{2.42}$$

We can identify

$$A_s = A_{11} - A_{12}P$$

$$A_f = A_{22} + PA_{12}.$$

Thus, we seek Q such that

$$A_sQ - QA_f + A_{21} = 0. \tag{2.43}$$

Again, this Lyapunov equation will have a unique solution since A_s and A_f have no eigenvalues in common. However, we can show that Q will be an

asymptotically stable equilibrium of the matrix difference equation

$$Q_{k+1} = A_{11}^{-1}(A_{12}PQ_k + Q_k(A_{22}+PA_{12})-A_{12}). \tag{2.44}$$

Using Lemma 2 of [6] we know that this equation will converge to a stable equilibrium if

$$c < \frac{1}{2}\,(a+b)^{-1}$$

which implies

$$\|A_{11}^{-1}\| < \frac{1}{2\mu}\,(\|\hat{A}_{22} + P_o\hat{A}_{12}\| + \|\hat{A}_{12}\|\,\|P_o\|)^{-1}$$

which is automatically satisfied by (2.38). Making the substitution $A_{ij} = \mu^{x}\hat{A}_{ij}$ in (2.44)

$$Q_{k+1} = \mu^{(1-j)}A_{11}^{-1}(\hat{A}_{12}Q_k + Q_k(\mu^j\hat{A}_{22} + P\hat{A}_{12})-\hat{A}_{12}).$$

Therefore

$$\begin{aligned} Q &\triangleq O(\mu^{1-j}) \triangleq M\big|_{j=0} \\ P &\triangleq O(\mu^{j}) \triangleq L\big|_{j=0}. \end{aligned} \tag{2.45}$$

If we had used the independent block transformation on the $x_1(k)$ subsystem first, (call this the "S" transformation)

$$\begin{aligned} y_1(k) &= x_1(k) + Px_2(k) \\ y_2(k) &= x_2(k) - Qy_1(k). \end{aligned} \tag{2.46}$$

A similar set of results will follow. We will outline these results to avoid too many repetitious derivations. The two matrix recursion equations are given as

$$\text{i.}\quad P_{k+1} = A_{11}^{-1}(A_{12} + P_kA_{22} - P_kA_{21}P_k) \tag{2.47}$$

$$P_o = A_{11}^{-1}A_{12}$$

$$\text{ii.}\quad Q_{k+1} = (A_{22}Q_k - A_{21}PQ_k - Q_kPA_{21} + A_{21})A_{11}^{-1} \tag{2.48}$$

$$Q_o = A_{21}A_{11}^{-1}$$

Both of these nonlinear matrix difference equations converge to stable equilibrium values if

$$\|A_{11}^{-1}\| < \frac{1}{3}\left(\|A_{22}-A_{21}P_o\| + \|P_o\|\ \|A_{21}\|\right)^{-1}.$$

Showing μ parameters explicitly,

$$\|A_{11}^{-1}\| < \frac{1}{3}\left(\|\mu\hat{A}_{22}-\mu^j\hat{A}_{21}A_{11}^{-1}\mu^{(1-j)}\hat{A}_{12}\| + \|\mu^j\hat{A}_{21}\|\ \|A_{11}^{-1}\mu^{(1-j)}\hat{A}_{12}\|\right)^{-1}$$

$$\|A_{11}^{-1}\| < \frac{1}{3\mu}\left(\|\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12}\| + \|\hat{A}_{21}\|\ \|A_{11}^{-1}\hat{A}_{12}\|\right)^{-1}$$

which gives us the following bound on μ

$$0 \le \mu < \frac{1}{\|A_{11}^{-1}\|\left(\|\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12}\| + \|\hat{A}_{21}\|\ \|A_{11}^{-1}\hat{A}_{12}\|\right)}. \tag{2.49}$$

Also, it is easily seen that

$$P \triangleq O(\mu^{1-j}) \qquad Q \triangleq O(\mu^j). \tag{2.50}$$

This similarity in bounds for transformations "F" and "S" is very important. Both of these transformations have been derived using our general two-time-scale form. Therefore, if μ satisfies both bounds, then either transformation can be used. To avoid confusion, in future sections the superscript s or f will be used in conjunction with the P and Q transformation matrices to distinguish which transformation was used.

2.6. Degenerate Cases

We defined a "degenerate" case as one where

$$1+\varepsilon < \theta+p < \infty \qquad 0 < \varepsilon \ll 1. \tag{2.51}$$

Let us take the degenerate model

$$\begin{bmatrix} y(k+1) \\ - - - - \\ z(k+1) \end{bmatrix} = \begin{bmatrix} A & \vdots & \mu^{\theta}B \\ - - & \vdots & - - \\ \mu^{\beta}C & \vdots & \mu D \end{bmatrix} \begin{bmatrix} y(k) \\ - - - \\ z(k) \end{bmatrix} \tag{2.52}$$

where $\hat{\theta}$ and $\hat{\beta}$ are bounded by (2.51).

Let us implicitly carry μ^{θ}, μ^{β}, and μ with B, C, and D respectively. Then, the sufficient condition for block diagonalization (transformation F) becomes

$$\|A^{-1}\| < \frac{1}{3} \left(\|D-CA^{-1}B\| + \|CA^{-1}\| \ \|B\|\right)^{-1}.$$

With explicit μ parameters we have

$$\|A^{-1}\| < \frac{1}{3} \left(\|\mu D-\mu^{\beta}CA^{-1}\mu^{\theta}B\| + \|\mu^{\beta}CA^{-1}\| \ \|\mu^{\theta}B\|\right)^{-1} \tag{2.53}$$

$$\|A^{-1}\| < \frac{1}{3\mu} \left(\|D-\mu^{(\theta+\beta-1)}CA^{-1}B\| + \mu^{\theta+\beta-1}\|CA^{-1}\| \ \|B\|\right)^{-1}.$$

Therefore

$$0 \leq \mu < \frac{1}{3\|A^{-1}\| \left(\|D-\mu^{(\theta+\beta-1)}CA^{-1}B\| + \mu^{\theta+\beta-1}\|CA^{-1}\| \ \|B\|\right)}. \tag{2.54}$$

However, if the system where "normal," the bound on μ would be

$$0 \leq \mu < \frac{1}{3\|A^{-1}\| \left(\|D-CA^{-1}B\| + \|CA^{-1}\| \ \|B\|\right)}.$$

We now state a lemma enabling us to show that the upper bound on μ for a degenerate case will always be greater than for a "normal" case. This will enable us to concentrate on "normal cases," realizing that by modeling the degenerate system as "normal" we will always achieve results in approximation, convergence, etc., that are equal to or better than the "normal" case.

Lemma 7: If a discrete-time system can fit our model only in a degenerate form, such that

$$1 \ll \theta+\beta < \infty$$

then the upper bound on μ needed for block diagonalization is greater than the corresponding normal case. In other words,

$$\frac{1}{3\|A^{-1}\|\,(\|D-\mu^{(\theta+\beta-1)}CA^{-1}B\| + \mu^{(\theta+\beta-1)}\|CA^{-1}\|\ \|B\|)} > \frac{1}{3\|A^{-1}\|\,(\|D-CAB\| + \|CA^{-1}\|\ \|B\|)} \tag{2.55}$$

$$\forall\ \theta+\beta > 1$$

Proof: (2.55) implies

$$\|D-CAB\| + \|CA^{-1}\|\ \|B\| > \|D-\mu^{\theta+\beta-1}CA^{-1}B\| + \mu^{\theta+\beta-1}\|CA^{-1}\|\ \|B\|$$

$$\|D-CA^{-1}B\| - \|D-\mu^{(\theta+\beta-1)}CA^{-1}B\| > (\mu^{\theta+\beta-1}-1)\|CA^{-1}\|\ \|B\|. \tag{2.56}$$

Let

$$F = \|D-CA^{-1}B\| - \|D-\mu^{(\theta+\beta-1)}CA^{-1}B\| \tag{2.57}$$

and let us examine the bounds of F as $\theta+p$ "swings" through the degenerate range. Then our inequalities become

$$\text{as}\quad \theta+\beta \to 1 \quad F \to 0 > (\mu^{\theta+\beta-1}-1)\|CA^{-1}\|\ \|B\| \tag{2.58}$$

$$\text{as}\quad \theta+\beta \to \infty \quad F \to \|CA^{-1}B\| > (\mu^{\theta+\beta-1}-1)\|CA^{-1}\|\ \|B\|. \tag{2.59}$$

Thus F is bounded such that

$$0 < F < \|CA^{-1}B\|$$

as long as $1 < \theta+\beta < \infty$, which defines a degenerate system. Therefore, if (2.55) is to be satisfied, then both upper and lower bounds on F must satisfy (2.56). Looking at (2.58)

$$0 > (\mu^{(\theta+\beta-1)}-1)\|CA^{-1}\|\ \|B\|$$

$$0 > \mu^{(\theta+\beta-1)} - 1$$

$$1 > \mu^{(\theta+\beta-1)} \tag{2.60}$$

which will be true for every degenerate value of $\theta+\beta$ since $\rightarrow 0<\mu \ll 1$. Just for completeness (2.59) gives

$$\|CA^{-1}B\| < (\mu^{\theta+\beta-1}-1)\|CA^{-1}\| \ \|B\|$$

$$\frac{\|CA^{-1}B\|}{\|CA^{-1}\| \ \|B\|} + 1 > \mu^{\theta+\beta-1}$$

which is always true if (2.60) is true. Therefore (2.55) holds.

The importance of this is that we may consider our general model as an "upper bound" for many degenerate systems which also satisfy a two-time-scale property. Also, since many of the models and algorithms proposed in this thesis are μ dependent in terms of accuracy and convergence properties, higher values of μ can be used with degenerate systems and still obtain good results. However, even more important, by modeling a degenerate system in our general form (i.e. B and C are no longer normalized to A), we can expect better results (convergence of algorithms, reduced order approximations, etc.) than for a similar system (same μ parameters) with "normal" submatrices B and C.

We can conclude this section by stating that by performing our analysis and design procedures on our "normal" general form, we can expect any degenerate form of this normal form to have more accurate results.

2.7. Duality of Transformations "S" and "F"

In two preceding sections sufficiency conditions for the block diagonalization of our general two-time-scale model and for the two limiting cases were carried out separately.

The purpose of this was to point out the dual nature of the transformations that were derived. In particular, transformation S may be considered the "slow" transformation since it will allow slow mode modification independent of the Q_k iteration (as shown in a later chapter). Likewise, F may be considered the "fast" transformation for the similar reason. This property will become very important when we consider eigenvalue placement.

When attempting to block diagonalize the general model, for either transformation we found very similar bounds on μ for every j including our limiting cases. What this means is that for μ small enough, either transformation may be used. Thus, we can choose whether we want our slow or fast state vector to be independent of Q_k.

Our two transformations in general form look like

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = S \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \qquad S = \left[\begin{array}{c|c} I-Q^sP^s & -Q^s \\ \hline P_s & I \end{array}\right] \tag{2.64}$$

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = F \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \qquad F = \left[\begin{array}{c|c} I & P^f \\ \hline -Q^f & I-Q^fP^f \end{array}\right] \tag{2.65}$$

where

$$Q^f \triangleq O(\mu^{(1-j)}) \qquad P^f \triangleq O(\mu^j)$$

$$Q^s \triangleq O(\mu^j) \qquad P^s \triangleq O(\mu^{(1-j)})$$

and the transformations are related by

$$S = (F^{-1})^T.$$

In block diagonalizing our limiting cases, $P^i \triangleq L \triangleq O(1)$, $i = s,f$ and $Q_i \triangleq M \triangleq O(\mu)$. This was done to preserve the order of L and M. However, for

$0 < j < 1$, the order of the matrices as well as the off diagonal system matrices will vary. When these transformations are independently applied to our general two-time-scale model, the resulting nonhomogeneous slow-fast systems are:

using "F"

$$\begin{bmatrix} x_s(k+1) \\ \hline x_f(k+1) \end{bmatrix} = \left[\begin{array}{c|c} (A_{11}-\mu\hat{A}_{12}P^f) & 0 \\ \hline 0 & \mu(\hat{A}_{22}+L\hat{A}_{12}) \end{array}\right] \begin{bmatrix} x_s(k) \\ \hline x_f(k) \end{bmatrix}$$

$$+ \begin{bmatrix} (I-Q^fP^f)B_1 - Q^fB_2 \\ \hline P^fB_1 + B_2 \end{bmatrix} u(k)$$

using "S"

$$\begin{bmatrix} x_s(k+1) \\ \hline x_f(k+1) \end{bmatrix} = \left[\begin{array}{c|c} (A_{11}+\mu Q^s\hat{A}_{21}) & 0 \\ \hline 0 & \mu(\hat{A}_{22}-\hat{A}_{21}P^s) \end{array}\right] \begin{bmatrix} x_s(k) \\ \hline x_f(k) \end{bmatrix}$$

$$+ \begin{bmatrix} B_1 + P^sB_2 \\ \hline -Q^sB_1 + (I-Q^sP^s)B_2 \end{bmatrix} u(k)$$

where $x_f(k)$ is independent of Q^f and $x_s(k)$ is independent of Q^s.

3. RECURSIVE PROPERTY AND DELAY APPROXIMATION

3.1. Introdution

This chapter serves primarily as an introduction to reduced order modeling of discrete two-time-scale systems fitting our general model. By taking advantage of the recursive property of discrete-time systems, we will be able to show how the $x_1(k+1)$ and $x_2(k+1)$ subsystems may be expressed as Nth order matrix difference equations in increasing powers of μ.

$$x_1(k+1) = M_1 x_i + \mu M_2 x_i(k-1) + \mu^N M_N x_i(k-N)$$

$$x_2(k+1) = K_1 x_i + \mu K_2 x_i(k-1) + \mu^N K_N x_i(k-N)$$

By truncating this "expansion" after, for example, two delays ($O(\mu^2)$), we will establish our first attempt at a reduced order "slow" model.

Then, since it is assumed that A_{11}^{-1} exists, we can take advantage of the backward recursion for the $x_1(k+1)$ subsystem,

$$x_1(k) = A_{11}^{-1} x_1(k+1) - A_{11}^{-1} \mu^{(1-j)} \hat{A}_{12} x_2(k) \qquad (3.1)$$

to "reflect" ahead the model of our system at some future iteration. This will result in an intuitively satisfying derivation of our slow subsystem found through transformation in Chapter 2. By continued application of the backwards recursion of $x_1(k)$ to our Nth order difference equations, the existence of our "P" transformation matrix will become apparent as we converge to a "steady state" or "slow" model.

We conclude this chapter with a look at the nonhomogeneous case to see how the control variables fit into this derivation.

Before going on to the next section, let us define a term that will be used throughout this chapter.

Definition

A matrix difference equation such that the next state, $x_i(k+1)$ is dependent on the previous N time instances such that

$$x_i(k+1) = f(x_j(k),\ x_j(k-1),\ldots,\ x_j(k-N)) \tag{3.2}$$

where j is not necessarily equal to i, will be refered to as an N-delay sequence of $x_i(k+1)$ in $x_j(k-j)$, $j = 0,1,2,\ldots N$.

3.2. Expansion of $x_1(k+1)$ Subsystem into an N-Delay Sequence

A linear shift-invariant discrete-time system is by nature a set of N 1-delay sequences. Thus, using the recursive property of discrete time systems, it can be expressed as a function of the last N events, a N-delay sequence. In this straightforward set of substitutions, we hope to show how, as more delay terms are added, there contribution to the next state $x_1(k+1)$ decreases as a power series in μ.

In the following, we will assume that the system is in our general linear shift-invariant discrete two-time-scale form. Also, for ease of notation, we will use the A_{ij} submatrices until the substitution $A_{ij} = \mu^{x}\hat{A}_{ij}$ is needed.

We now hope to show that for any system that can be put into our general two-time scale form, the $x_1(k)$ states can be approximated to $0(\mu^N)$ by an autonomous (function of only delayed versions of itself) delay-N sequence.

Given

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} \tag{3.3}$$

where i. $x_1(k) \in R^{N_s}$, $x_2(k) \in R^{N_f}$

ii. A_{ij}, $j = 1,2$ are real constant coefficient matrices satisfying our general two-time-scale structure

$$x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k)$$

$$= A_{11}x_1(k) + A_{12}A_{21}x_1(k-1) + A_{12}A_{22}x_2(k-1)$$

continued substitution of

$$x_2(k-j) = A_{21}x_1(k-(j+1)) + A_{22}x_2(k-1(j+1)) \tag{3.4}$$

results in

$$x_1(k+1) = A_{11}x_1(k) + A_{12}A_{21}x_1(k-1) + A_{12}A_{22}A_{21}x_1(k-2) + \ldots +$$

$$A_{12}A_{22}^{N-1}A_{21}x_1(k-N) + A_{12}A_{22}^{N}x_2(k-N)$$

Now, making the substitutions $A_{ij} = \mu^{x}\hat{A}_{ij}$,

$$x_1(k+1) = A_{11}x_1(k) + \mu^{(i-j)}\hat{A}_{21}x_1(k-1)$$

$$+ \mu^{(i-1)}\hat{A}_{12}\mu\hat{A}_{22}\mu^{j}\hat{A}_{21}x_1(k-2)$$

$$+ \ldots + \mu^{(i-j)}\hat{A}_{12}(\mu\hat{A}_{22})^{N-1}\mu^{j}\hat{A}_{21}x_1(k-N)$$

$$+ \mu^{(i-j)}\hat{A}_{12}(\mu\hat{A}_{22})^{N}x_2(k-N) \tag{3.5}$$

$$= A_{11}x_1(k) + \mu\hat{A}_{12}\hat{A}_{21}x_1(k-1) + \mu^2\hat{A}_{12}\hat{A}_{22}\hat{A}_{21}x_1(k-2)$$

$$+ \ldots + \mu^N\hat{A}_{12}\hat{A}_{22}^{N-1}\hat{A}_{21}x_1(k-N) + \mu^{N+1-j}\hat{A}_{12}\hat{A}_{22}^{N}x_2(k-N) \quad (3.6)$$

The significance of (3.6) is that for small perturbations in $x_2(0)$, the entire dynamics of the $x_1(k)$ subsystem may be represented accurately to $0(\mu^N)$ by an N-delay sequence in $x_1(k)$.

Thus, by using the process of state augmentation, we can simulate the $x_1(k)$ states to a desired accuracy.

For example, if we want $0(\mu^2)$ accuracy, our model would look like

$$x_1(k+1) = A_{11}x_1(k) + \mu\hat{A}_{12}\hat{A}_{21}x_1(k-1)$$

Let $z(k) = x_1(k-1)$

$$\begin{bmatrix} x_1(k+1) \\ \hline z(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & \mu\hat{A}_{12}A_{21} \\ \hline I & 0 \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} \quad (3.7)$$

where $x_1(0) = x_{10}$, $z(0) = 0$.

For $N_S \ll N_F$ this model is good for μ sufficiently small. However, for $N_S \approx N_F$ or for μ too large, the order of the model can surpass the actual system model to achieve reasonable accuracy. As a first attempt, this model proved to be very important as will be seen in future sections.

3.3. Use of Backward Recursion to Establish Steady State Models

In the previous section, the backward recursion (3.1) of $x_1(k)$ was never used in obtaining our Nth order difference equation for $x_1(k+1)$ (3.6). In this section, however, the general form of (3.1),

$$x_1(k-j) = A_{11}^{-1}x_1(k-j+1) - A_{11}^{-1}\mu^{(i-j)}\hat{A}_{12}x_2(k-j)$$

$$= j = 0,1,2,\ldots\ldots \qquad (3.8)$$

is applied once, and then N times at each iteration to obtain "steady state" models of the system at the next or Nth iteration.

Again, given system (3.3),

$$x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) \qquad (3.9)$$

$$x_2(k) = A_{21}x_1(k-1) + A_{22}x_2(k-1) \qquad (3.10)$$

(3.10) into (3.9) gives

$$x_1(k+1) = A_{11}x_1(k) + A_{12}A_{21}x_1(k-1) + A_{12}A_{22}x_2(k-2) \qquad (3.11)$$

using (3.8) for j = 1, into (3.11)

$$x_1(k+1) = A_{11}x_1(k) + A_{12}A_{21}A_{11}^{-1}x_1(k)$$

$$+ A_{12}(A_{22}-A_{21}A_{11}^{-1}A_{12})x_2(k-1)$$

$$= (A_{11} + A_{12}A_{21}A_{11}^{-1})x_1(k)$$

$$+ A_{12}(A_{22}-A_{21}A_{11}^{-1}A_{12})x_2(k-1) \qquad (3.12)$$

now, into (3.12)

$$x_2(k-1) = A_{21}x_1(k-2) + A_{22}x_2(k-2)$$

$$= [A_{22}-A_{21}A_{11}^{-1}A_{12}]x_2(k-2) + A_{21}A_{11}^{-1}x_1(k-1)$$

$$x_1(k+1) = (A_{11}+A_{12}A_{21}A_{11}^{-1})x_1(k)$$

$$+ (A_{12}(A_{21}-A_{21}A_{11}^{-1}A_{12})A_{21}A_{11}^{-1})x_1(k-1)$$

$$+ (A_{12}(A_{22}-A_{21}A_{11}^{-1}A_{12})^2)x_2(k-2) \qquad (3.13)$$

Continue these substitutions in the same manner using the general equation

$$x_2(k-j) = (A_{22}-A_{21}A_{11}^{-1}A_{12})x_2(k-j-1) + A_{21}A_{11}^{-1}x_1(k-j)$$

$$j = 0,1,..... \qquad (3.14)$$

We generate the following delay-N sequence.

$$x_1(k+1) = (A_{11} + A_{12}A_{21}A_{11}^{-1})x_1(k)$$

$$+ (A_{12}(A_{22}-A_{21}A_{11}^{-1}A_{12})A_{21}A_{11}^{-1})x_1(k-1)+....$$

$$+ (A_{12}(A_{22}-A_{21}A_{11}^{-1}A_{12})^{N-1}A_{21}A_{11}^{-1})x_1(k(N-1))$$

$$+ (A_{12}(A_{22}-A_{21}A_{11}^{-1}A_{12})^{N})x_2(k-N) \qquad (3.15)$$

Again, making the substitutions $A_{ij} = \mu^{x}\hat{A}_{ij}$

$$x_1(k+1) = (A_{11} + \mu^{(1-j)}\hat{A}_{12}\mu^{j}\hat{A}_{21}A_{11}^{-1})x_1(k)$$

$$+ (\mu^{(1-j)}\hat{A}_{12}(\mu\hat{A}_{22}-\mu^{j}\hat{A}_{21}A_{11}^{-1}\mu^{(1-j)}\hat{A}_{12})\mu^{j}\hat{A}_{21}A_{11}^{-1})x_1(k-1)$$

$$+....+$$

$$+ (\mu^{(1-j)}\hat{A}_{12}(\mu\hat{A}_{22}-\mu^{j}\hat{A}_{21}A_{11}^{-1}\mu^{(1-j)}\hat{A}_{12})^{N-1}\mu^{j}\hat{A}_{21}A_{11}^{-1})x(k-(N-1))$$

$$+ (\mu^{(1-j)}\hat{A}_{12}(\mu\hat{A}_{22}-\mu^{j}\hat{A}_{21}A_{11}^{-1}\mu^{(1-j)}\hat{A}_{12})^{N})x_2(k-N) \qquad (3.16)$$

since μ, μ^{j}, μ^{1-j} are positive scalers, they can be combined to give

$$x_1(k+1) = (A_{11}+\mu\hat{A}_{12}\hat{A}_{21}A_{11}^{-1})x_1(k)$$

$$+\mu^2(\hat{A}_{12}(\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12})\hat{A}_{21}A_{11}^{-1})x_1(k-1)$$

$$+.....$$

$$+\mu^N(\hat{A}_{12}(\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12})^{N-1}\hat{A}_{21}A_{11}^{-1})x_1(k-(N-1))$$

$$+\mu^{N+1-j}(\hat{A}_{12}(A_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12})^{N})x_2(k-N) \qquad (3.17)$$

In arriving at (3.17) it is important to understand that we have applied the backward recursion (3.8) <u>once</u> for each <u>delayed</u> term in (3.6). This substitution has "reflected ahead" the influence of the $x_1(k-M)^{th}$ term on the $x_1(k-(M-1))^{th}$ term. Conceptually, you may look at (3.17) as our N-delay sequence model of our two-time-scale system for $k \geq 1$.

In comparing N-delay sequence models (3.6) and (3.17), we notice that every term on the right hand side of (3.6) is of the form

$$\mu^M f(x_1-M) \quad M \geq 1$$

where, the equivalent delayed term of (3.17) is given as

$$\mu^{M+1} g(x_1-M) \quad M \geq 1$$

The homogeneous part of (3.17) (M = 0) has 0(1) and $0(\mu)$ term whereas for (3.6), the homogeneous part possesses only an 0(1) term.

In Chapter 2, through transformation we arrived at a slow subsystem matrix A_s. For the "F" transformation this matrix was composed of

$$A_s = A_{11} - A_{12}P_k$$

$$P_{k+1} = (A_{22}P_k + P_kA_{12}P_k - A_{21})A_{11}^{-1} \quad P_o = -A_{21}A_{11}^{-1}$$

For $k = 0$

$$A_s = A_{11} + A_{12}A_{21}A_{11}^{-1}$$

and using $A_{ij} = \mu^{x}\hat{A}_{ij}$

$$A_s = A_{11} + \mu\hat{A}_{12}\hat{A}_{21}A_{11}^{-1} \tag{3.18}$$

Notice that (3.18) is just the homogeneous part of (3.17). So, by substituting the backwards recursion into (3.6) at each delayed term ($M \geq 1$), we have generated the model of the system at the next iteration. Since $x_1(k+1)$ does represent our slow state variable, the equivalence of the homogeneous part of (3.17) and equation (3.18) seem logical.

It is of interest now to examine truncated models of (3.6) and (3.17) so as to determine which would be more a accurate model of the $x_1(k)$ states. Three criteria must be examined in making a judgement.

1) Number of matrix multiplications required to generate the model
2) Order of model
3) Magnitude of μ

For an equal number of multiplications, (3.6) looks like

$$x_1(k+1) = A_{11}x_1(k) + \mu\hat{A}_{12}\hat{A}_{21}x_1(k-1) \tag{3.19}$$

and (3.17) like

$$x_1(k+1) = [A_{11} + \mu\hat{A}_{12}\hat{A}_{21}A_{11}^{-1}]x_1(k) \tag{3.20}$$

However, (3.19) is twice the order of (3.20). For small μ, (3.19) and (3.20) will give approximately the same accuracy. Therefore, since lower order is usually preferred (3.20) would be the logical choice.

As μ becomes larger, the $0(\mu)$ terms in (3.19) and (3.20) become more significant. Then, the way (3.20) is derived from (3.19) becomes important. In other words,

Given

$$x_1(k+1) = A_{11}x_1(k) + \mu\hat{A}_{12}\hat{A}_{21}x_1(k-1)$$

use

$$x_1(k-1) = A_{11}^{-1}x_1(k) - \mu^{(1-j)}A_{11}^{-1}\hat{A}_{12}x_2(k-1)$$

This gives,

$$x_1(k+1) = A_{11}x_1(k) + \mu\hat{A}_{12}\hat{A}_{21}[A_{11}^{-1}x_1(k) - \mu^{(1-j)}A_{11}^{-1}\hat{A}_{12}x_2(k-1)]$$

$$= [A_{11} + \mu\hat{A}_{12}\hat{A}_{21}A_{11}^{-1}]x_1(k) - \mu^{2-j}\hat{A}_{12}\hat{A}_{21}A_{11}^{-1}\hat{A}_{12}x_2(k-1) \tag{3.21}$$

Equation (3.20) neglects the $0(\mu^{2-j})$ term which could be significant for larger μ. When this is the case, (3.19) will give a more accurate model. This method of modeling will be refered to as "delay compensation", since we are using delays to compensate for nonneglible fast modes (larger μ). More will be said about this.

Now, using this same line of thought, let us apply (3.8) N times at each delay term in (3.6) for $M \geq 1$.

If the backward recursion (3.8) is repeatedly applied to (3.6)

until f_o has $0(\mu^N)$ accuracy where,

$$x_1(k+1) = f_0 x_1(k) + f_1 x_1(k-1) + \ldots + f_N x_1(k-M) + g_N x_2(k-(N+1))$$

Then $f_o = A_s + 0(\mu^N)$

and $f_1 \triangleq f_2 \triangleq \ldots \triangleq f_N \triangleq g_N = 0(\mu^N)$

To properly show this using induction is straightforward but very tedious.

Here, we offer a simple heuristic explanation merely expanding the series to N = 4. Given our original system equations,

$$x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k)$$

$$x_2(k+1) = A_{21}x_1(k) + A_{22}x_2(k)$$

using the following substitutions,

$$x_2(k-j) = (A_{22} - A_{21}A_{11}^{-1}A_{12})x_2(k-(j-1)) + A_{21}A_{11}^{-1}x_1(k-j)$$

$$x_1(k-j) = A_{11}^{-1}x_1(k-(j+1)) - A_{11}^{-1}A_{12}x_2(k-j) \quad \forall_j$$

for N = 1,

$$x_1(k+1) = [A_{11} + A_{12}A_{21}A_{11}^{-1}]x_1(k) + A_{12}[A_{22} - A_{21}A_{11}^{-1}A_{12}]x_2(k-1)$$

$$= [A_s + 0(\mu)]x_1(k) + 0(\mu)x_2(k-1)$$

for N = 4,

$$x_1(k+1) = [A_{11} + A_{12}A_{21}A_{11}^{-1} + A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})A_{21}A_{11}^{-2}$$

$$- A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})A_{21}A_{11}^{-2}A_{12}A_{21}A_{11}^{-2} + A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^2$$

$$A_{21}A_{11}^{-3}]x_1(k)$$

$$- A_{12}[A_{22} - A_{21}A_{11}^{-1}A_{12}]^2A_{21}A_{11}^{-3}A_{12}A_{21}A_{11}^{-1}x_1(k-1)$$

$$- A_{12}[A_{22} - A_{21}A_{11}^{-1}A_{12}]^2A_{21}A_{11}^{-1}A_{12}A_{21}A_{11}^{-1}x_1(k-2)$$

$$\left[+ A_{12}[A_{22} - A_{21}A_{11}^{-1}A_{12}]A_{21}A_{11}^{-2}A_{12}A_{21}A_{11}^{-2}A_{12}A_{21}A_{11}^{-1} \right.$$

$$\left. + A_{12}[A_{22} - A_{21}A_{11}^{-1}A_{12}]^3A_{21}A_{11}^{-1} \right] x_1(k-3)$$

$$+ O(\mu^4) \qquad (3.22)$$

Which is of the form

$$x_1(k+1) = f_o x_1(k) + f_1 x_1(k-1) + f_2 x_1(k-2) + f_3 x_1(k-3)$$

where $f_o = A_{11} - A_{12}P_k = A_s + O(\mu^4)$

and $P_{k+1} = (P_k A_{12} P_k + A_{22} P_k - A_{21})A_{11}^{-1} \quad k = 0,1,2$

$P_o = -A_{21}A_{11}^{-1}$

and from (3.22) it is obvious that f_1, f_2, f_3 and $g_3 \triangleq O(\mu^4)$.

Continuing in this manner results in the P_k transformation matrix recursion of Chapter 2. Conditions for its convergence have already been studied.

Remark

The recursive substitutions were made in an attempt to converge to the F transformation matrices. By performing the dual substitutions, the S transformation matrices could have been easily obtained.

Thus, we have shown that by applying the backward recursion equation (3.8) repeatedly to the N-delay sequence of $x_1(k+1)$ (3.6), we approach a "steady-state" or slow model of the system at some future iteration N. We identified this with our slow subsystem developed through transformation in Chapter 2. For μ small, this "steady-state" or slow model will prove to be very accurate in approximating the $x_1(k)$ states. For μ larger, the initial deviation of $x_1(k)$ due to fast modes may be significant. "Delay compensation" may then be used to more closely model these initial deviations. We refer this initial period of deviation as the "boundary layer"[2]. The existence of this "layer" will become apparent when we next discuss the $x_2(k+1)$ subsystem.

3.4. $x_2(k+1)$ Expansion and the Boundary Layer

Up to now, there has been no attempt to simulate the $x_2(k+1)$ subsystem. If the same autonomous expansion is attempted such that

$$x_2(k+1) = f(x_2(k),\ x_2(k-1),\ldots\ x_2(k-N))$$

we get,

$$x_2(k+1) = A_{22}x_2(k) + A_{21}A_{12}x_2(k-1)$$

$$+ A_{21}A_{11}A_{12}x_2(k-2) + \ldots\ldots$$

$$+A_{21}A_{11}^{N-1}A_{12}x_2(k-N)+A_{21}A_{11}^{N}x_1(k-N)$$

making the substitution $A_{ij}=\mu^{x}\hat{A}_{ij}$

$$\begin{aligned}x_2(k+1)=&\mu\hat{A}_{22}x_2(k)+\mu^{j}\hat{A}_{21}\mu^{(1-j)}\hat{A}_{12}x_2(k-1)\\&+\mu^{(1-j)}\hat{A}_{21}A_{11}\mu^{j}\hat{A}_{12}x_2(k-2)+\ldots\\&+\mu^{j}\hat{A}_{21}A_{11}^{N-1}\mu^{(1-j)}\hat{A}_{12}x_2(k-N)\\&+\mu^{j}\hat{A}_{21}A_{11}^{N}x_1(k-N)\end{aligned}\tag{3.23}$$

$$\begin{aligned}x_2(k+1)=&\mu\hat{A}_{22}x_2(k)+\mu\hat{A}_{21}\hat{A}_{12}x_2(k-1)\\&+\mu\hat{A}_{21}A_{11}\hat{A}_{12}x_2(k-2)+\ldots.\\&+\mu\hat{A}_{21}A_{11}^{N-1}\hat{A}_{12}x_2(k-N)\\&+\mu^{j}\hat{A}_{21}A_{11}^{N}x_1(k-N)\end{aligned}\tag{3.24}$$

Examining (3.24), we see the exists <u>no</u> truncated delay sequence which could accurately model the $x_2(k)$ states. If fact, the $x_1(k-N)$ term will always be dominant. Thus, the lowest order simulation model for the $x_2(k)$ states is just the original state equation

$$x_2(k+1)=\mu^{j}\hat{A}_{21}x_1(k)+\mu\hat{A}_{22}x_2(k)$$

which is obviously nonautonomous and therefore does us no good.

Now, if we try to derive the N-delay sequence for $x_2(k+1)$ such that

$$x_2(k+1) = f(x_1(k),\ x_1(k-1),\ldots,\ x_1(k-N))$$

we get

$$x_2(k+1) = A_{21}x_1(k) + A_{22}A_{21}x_1(k-1)$$
$$+ A_{22}^2A_{21}x_1(k-2)+\ldots+ A_{22}^NA_{21}x_1(k-N)$$
$$+ A_{22}^{N+1}x_2(k-N)$$

Substituting $A_{21} = \mu^j\hat{A}_{21}$, $A_{22} = \mu\hat{A}_{22}$, $A_{12} = \mu^{(1-j)}\hat{A}_{12}$

$$x_2(k+1) = \mu^j\hat{A}_{21}x_1(k) + \mu^{1+j}\hat{A}_{22}\hat{A}_{21}x_1(k-1)$$
$$+ \mu^{2+j}\hat{A}_{22}\hat{A}_{21}x_1(k-2)+\ldots+\mu^{N+j}\hat{A}_{22}\hat{A}_{21}x_1(k-N)$$
$$+ \mu^{N+1}\hat{A}_{22}x_2(k-N) \qquad (3.25)$$

Again, we have a case similar to the $x_1(k+1)$ subsystem. We can totally simulute the $x_2(k+1)$ subsystem to $0(\mu^{N+j})$ accuracy using a truncated N-delay sequence. For $N \geq 1$, using (3.25) would result in "delay compensation" terms implimented using state augmentation which has been already discussed.

One very noticible difference between the $x_2(k+1)$ sequence and the $x_1(k+1)$ sequence is the preservation of initial conditions. When we truncate (3.25) we eliminate any dependence of $x_2(k+1)$ on $x_2(0)$. Thus, from $k = 0$ to $k = k^*$, (3.25) may differ considerable from the actual $x_2(k+1)$ states. Since all fast modes are stable in this analysis, the question is not whether the truncated version of (3.25), $\bar{x}_2(k+1)$, will converge to $x_1(k+1)$, but <u>how soon</u>

$$|\bar{x}_2(k+1) - x_2(k+1)| < \epsilon \quad \text{for some } \epsilon > 0$$

$$\text{and any } k \geq 0$$

The period of time $[0,k^*]$ is refered to as the "boundry layer" in the analysis of continuous systems possessing a similar property [2,4]. For μ small, k^* can be as small as 1. In this case, only a few terms of (3.25) would be needed. However, as μ becomes larger, an impractical number of terms would be needed for simulation.

Now, let us apply the backwards recursion of (3.6) to (3.25), so as to get $0(\mu^{N+j})$ accuracy for each delay term. After considerable algebraic manipulation, (3.25) becomes of the form

$$x_2(k+1) = f_0 x_1(k) + f_1 x_1(k-1) + f_2 x_1(k-2) + \ldots + f_N x_1(k-N)$$

where

$$f_o = (A_{21}A_{11}^{-1} + A_{22}A_{21}A_{11}^{-1} - A_{21}A_{11}^{-1}A_{12}A_{21}A_{11}^{-1} + \ldots)$$

$$(A_{11} + A_{12}A_{21}A_{11}^{-1} + \ldots.) \qquad (3.26)$$

$$= -(P_N)(A_S)x_1(k)$$

$$f_1 \triangleq f_2 \triangleq f_3 \ldots \triangleq f_N = 0(\mu^{N+j})$$

where P_N is the Nth iteration of

$$P_{k+1} = (P_k A_{12} P_k + A_{22} P_k - A_{21})A_{11}^{-1}$$

$$P_o = -A_{21}A_{11}^{-1} \qquad k = 0,1,\ldots.N$$

and

$$A_S = A_{11} - A_{12}P_k$$

The similarity between these expansions and the "F" transformation should now be completely clear. This transformation is given again for convenience

$$x_1(k) = x_s(k) + Q^F x_f(k)$$

$$x_2(k) = -P^F x_s(k) + (I - P^F Q^F) x_f(k)$$

By applying the backwards recursion (3.8) on (3.6) and (3.25), we are finding our system model at some future iteration N. In a two-time-scale system, the farther N is away from k = 0, the smaller the $x_f(k)$ state vector will be. In fact, it will have decayed to $0(\mu^N)$ proportion.

This therefore verifies,

$$x_1(k+1) = A_s x_s(k) + 0(\mu^N)$$

$$x_2(k+1) = -P^F A_s x_s(k) + 0(\mu^N) \quad k \geq N$$

for our N-delay sequences.

So, for μ sufficiently small, we can approximate $x_1(k+1)$ by

$$x_1(k+1) = A_s x_1(k) \tag{3.27}$$

and the $x_2(k+1)$ states as a function of $x_1(k+1)$ states $\forall k \geq 0$

$$x_2(k+1) = -P^F A_s x_s(k) = -P^f x_1(k+1) \tag{3.28}$$

or just an output variable.

However, if we use (3.27) and (3.28) and μ is large, these approximations may deviate considerably during the boundry layer. To improve the boundary layer approximations, delay compensation terms may be added (i.e. $x_1(k-j)$). In this case, the delay term with maximum j terminates all backwards recursion substitutions instead of the $x_1(k)$ term.

In the next section we will give an example. Then, in the following section we will see how the control variable $\underline{u}(k)$ would enter into our N-delay sequence for $x_1(k+1)$ states. We will limit this discussion to the $x_1(k+1)$ subsystem since the "steady-state" dynamic of $x_2(k+1)$ states will be determined by the $x_1(k+1)$ states.

3.5. Example of Reduced Order Modeling

The model of a steam power system discussed in [19] is given as

$$\dot{z} = \begin{bmatrix} -2 & 0 & 0 & 0 & -4 \\ 4.75 & -5 & 0 & 0 & 0 \\ 0 & .167 & -.167 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & .025 & .02333 & .035 & -.1125 \end{bmatrix} z$$

To put this system into singularly perturbed form we first re-index the state variables using the permutation

$$P_1 = (e_4, e_5, e_2, e_3, e_1)$$

Where e_i is an elementary column vector whose ith entry is 1. With $y = Pz$, the new system becomes

$$\dot{y} = \begin{bmatrix} -.1125 & .02333 & .035 & 0 & .025 \\ 0 & -.167 & 0 & 0 & .167 \\ 0 & 2 & -2 & 0 & 0 \\ -4 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 4.75 & -5 \end{bmatrix} y$$

Now, we scale the states using $x = Sy$, where $S = \text{diag}(4,1,1,2,1)$. The resulting system is

$$\dot{x} = \begin{bmatrix} -.1125 & .0932 & .14 & 0 & .1 \\ 0 & -.167 & 0 & 0 & .167 \\ 0 & 2 & -2 & 0 & 0 \\ -2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2.375 & -5 \end{bmatrix} x$$

which is of the proper form in accordance with [5].

If this system is sampled at .8 second intervals, the discrete model is

$$x(k+1) = \begin{bmatrix} .9014 & .1179 & .0525 & .0167 & .02104 \\ -.0196 & .8743 & 0 & .025 & .02934 \\ -.0071 & .7342 & .20175 & .013 & .021067 \\ -.75 & -.0557 & -.032 & .19357 & -.014076 \\ -.306 & -.01694 & -.011 & .14278 & .013217 \end{bmatrix} x(k)$$

with eigenvalues

$$\lambda_{1,2} = .8777 \pm .1054j$$

$$\lambda_3 = .0179$$

$$\lambda_{4,5} = .2055 \pm .0236j$$

therefore $N_s = 2$, $N_f = 3$

$$\|A_{11}\| = .9415, \quad \|A_{12}\| = .0625, \quad \|A_{21}\| = .8184$$

$$\|A_{22}\| = .2441$$

The system is degenerate with

$$\mu = .259253$$

which is less than its defined upper bound of .27158. Using one iteration of our P^f successive approximation

$$P_1^f = \begin{bmatrix} -.025827 & -1.029 \\ 1.0207 & -.09667 \\ .4817 & -.06101 \end{bmatrix}$$

we arrive at our slow subsystem

$$x_s(k+1) = \begin{bmatrix} .875521 & .1748599 \\ -.059322 & .8779818 \end{bmatrix} x_s(k)$$

with eigenvalues

$$\lambda_{1,2} = .87675 \pm .10184j$$

We now wish to model the entire system dynamics using.

$$x_1(k+1) = A_s x_1(k) \quad x_1(k) \in R^2$$

Where the remaining three states appear as output functions or

$$x_2(k) = -P_1^f x_1(k) \quad x_2(k) \in R^3$$

To compare this second order model with our fifth order model, the response of the system to an initial perturbation of

$$x_o = (1, -.8, .5, .2, .6)$$

is plotted versus its lower order approximation (Figures 3.1 to 3.5). The actual (high order) states will be designated $x_i(k)$, while the approximated state will be identified by $\hat{x}_i(k)$.

As you can see, using just <u>one</u> iteration of P_k^f result in very good accuracy.

3.6. Nonhomogeneous Case: Delay-N Sequence

Given:

$$\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k) \\ x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k) \end{aligned} \tag{3.29}$$

where our μ parameters are again assumed implicit.

In this section we will simplify the derivations of section 3.1 and briefly describe how the appearance of a control u(k) will fit in to the delay-N sequence expansion of a general two-time-scale discrete system.

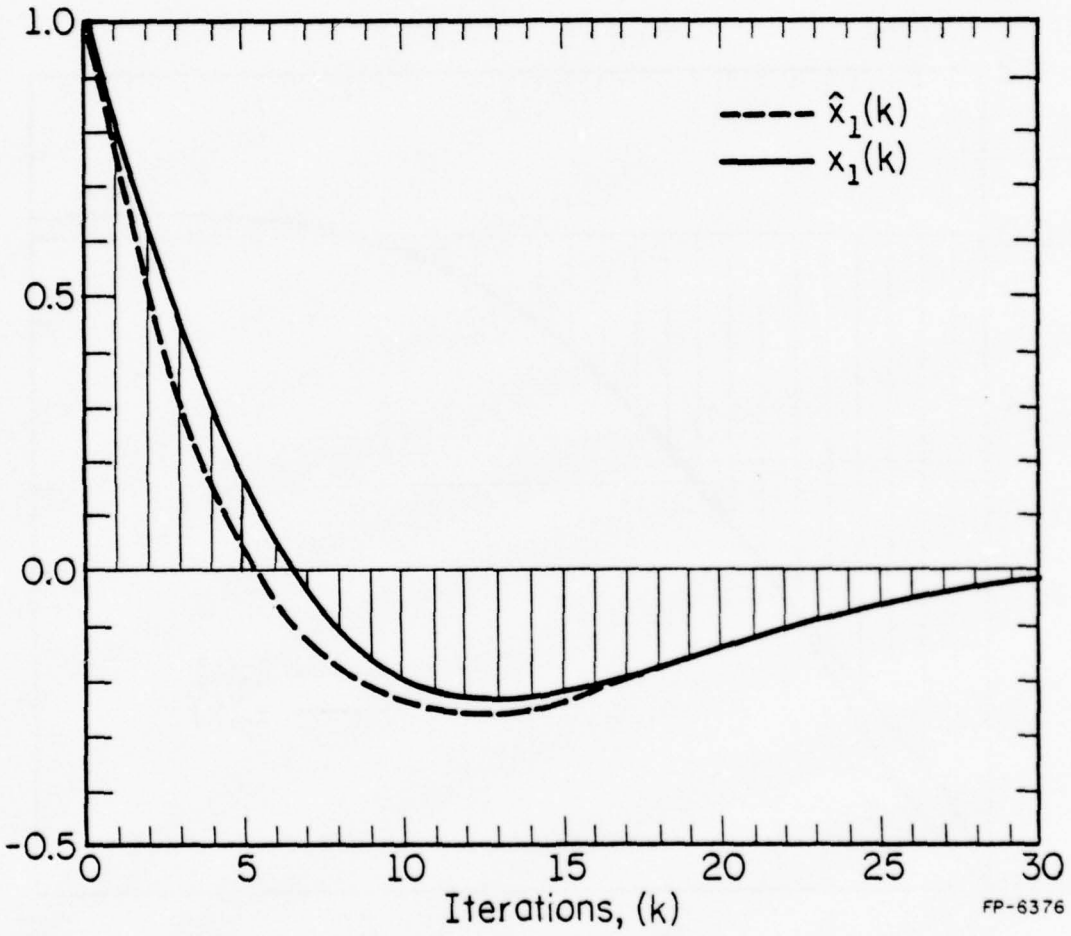

Figure 3.1. Reduced order approximation of state $x_1(k)$.

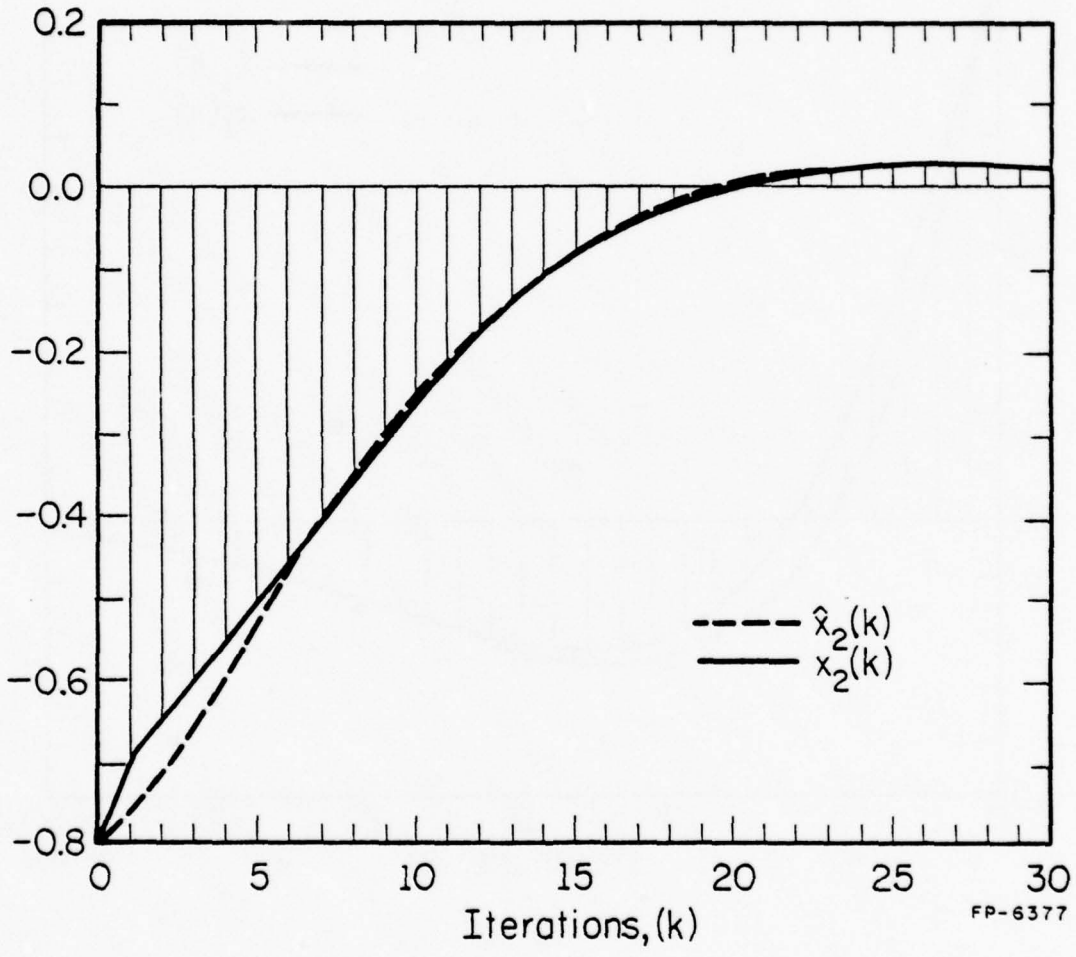

Figure 3.2. Reduced Order Approximation of state $x_2(k)$.

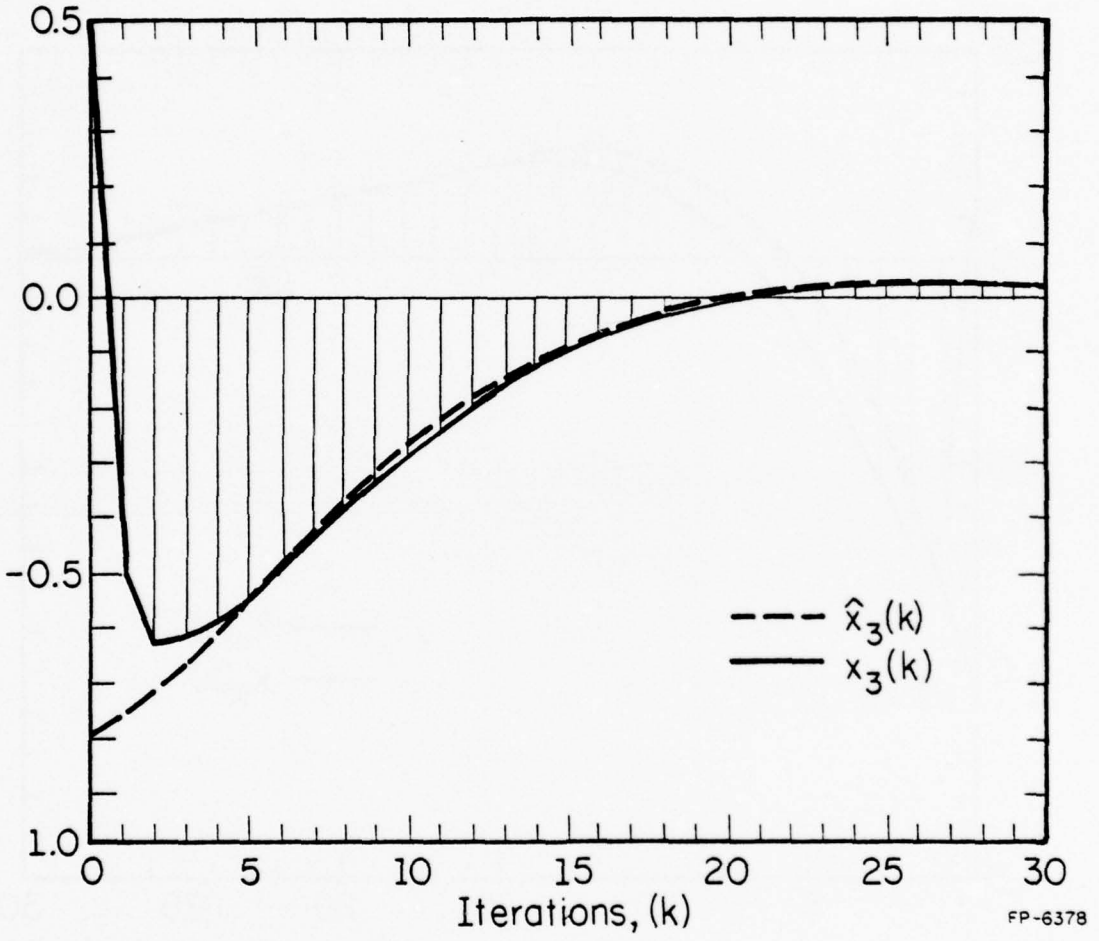

Figure 3.3. Reduced order approximation to state $x_3(k)$.

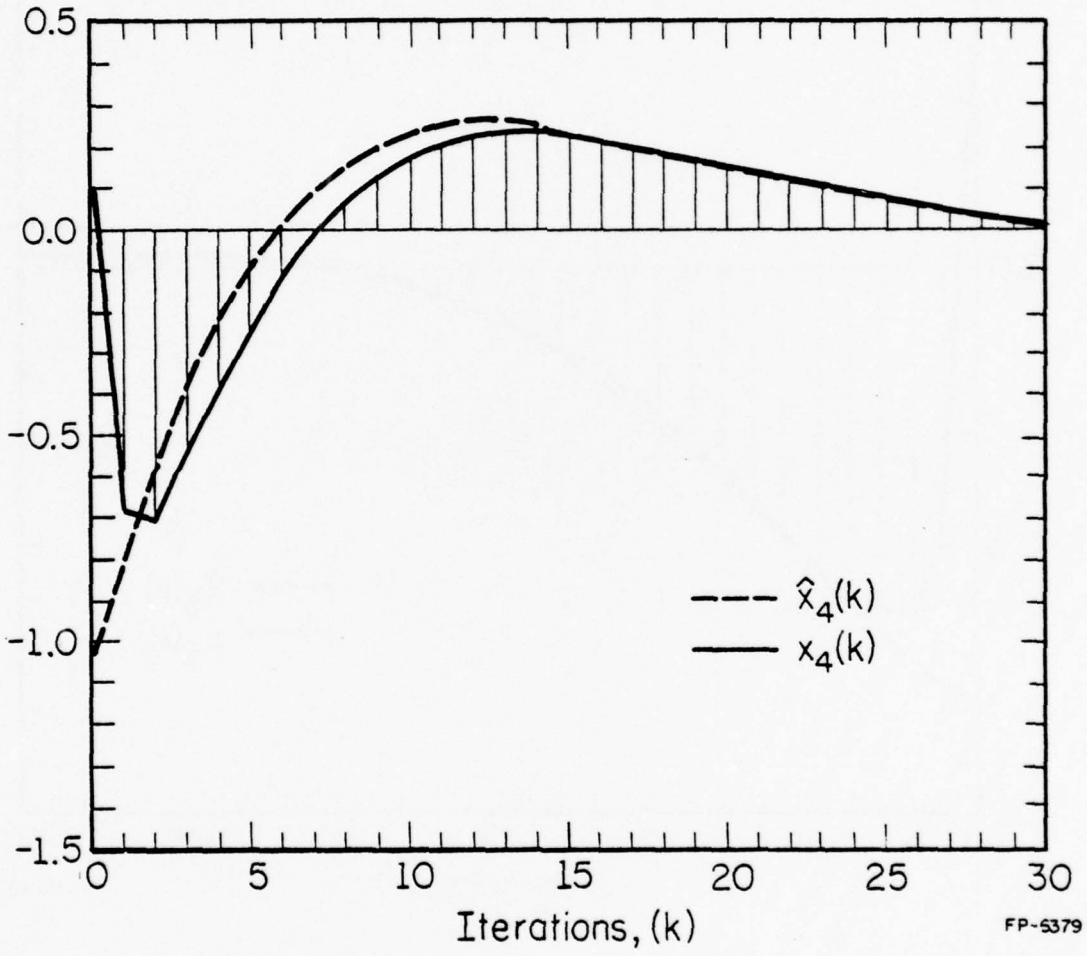

Figure 3.4. Reduced order approximation of state $x_4(k)$.

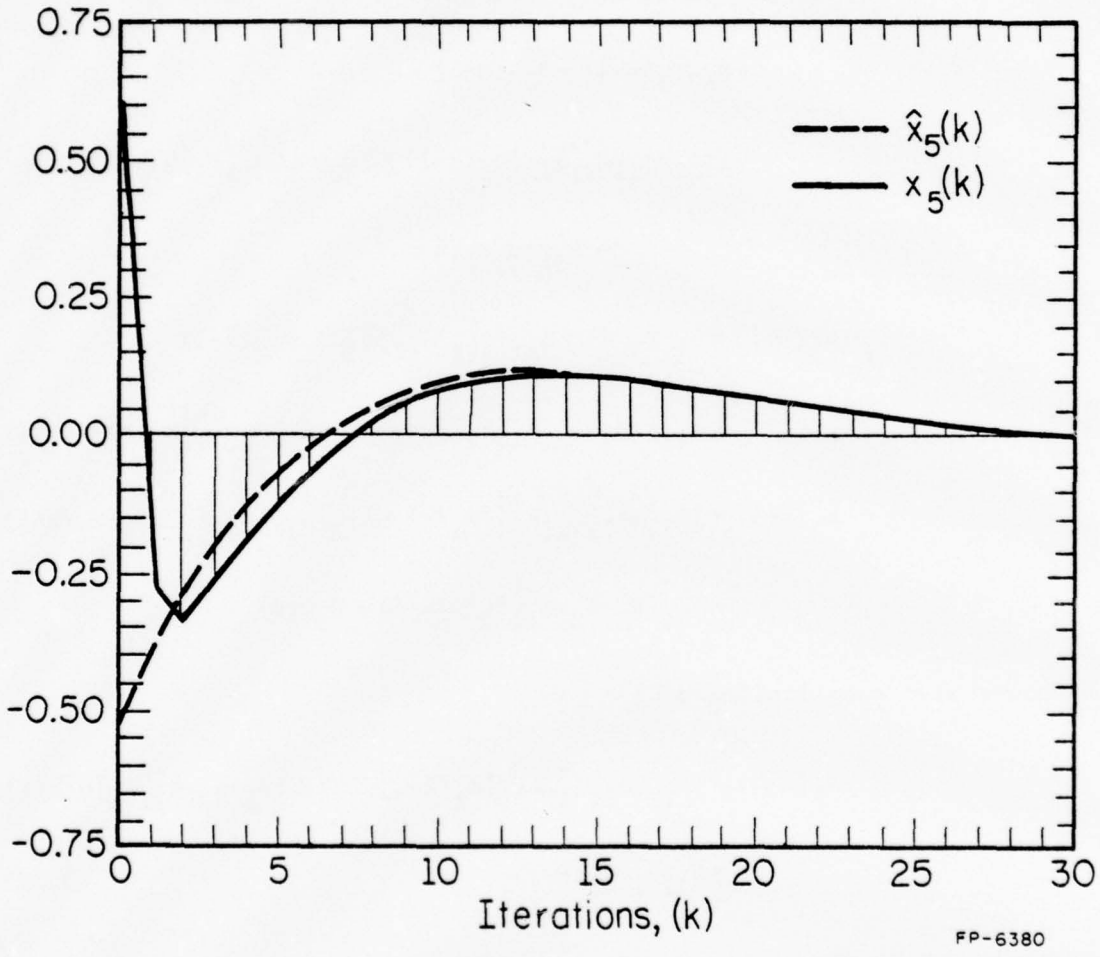

Figure 3.5. Reduced order approximation of state $x_5(k)$.

From 3.29

$$x_2(k) = A_{21}x_1(k-1) + A_{22}x_2(k-1) + B_2u(k-1)$$

$$x_1(k-1) = A_{11}^{-1}x_1(k) - A_{11}^{-1}A_{12}x_2(k-1) - A_{11}^{-1}B_1u(k-1)$$

$$x_2(k) = A_{21}A_{11}^{-1}x_1(k) - A_{21}A_{11}^{-1}A_{12}x_2(k-1) - A_{21}A_{11}^{-1}B_1u(k-1)$$

$$+ A_{22}x_2(k-1) + B_2u(k-1)$$

$$= [A_{22} - A_{21}A_{11}^{-1}A_{12}]x_2(k-1) + [B_2 - A_{21}A_{11}^{-1}B_1]u(k-1)$$

$$+ A_{21}A_{11}^{-1}x_1(k)$$

$$x_2(k+1) = A_{11}x_1(k) + A_{12}[A_{22} - A_{21}A_{11}^{-1}A_{12}]x_2(k-1)$$

$$+ A_{12}[(B_2 - A_{21}A_{11}^{-1}B_1)]u(k-1) + A_{12}A_{21}A_{11}^{-1}x_1(k) + B_1u(k)$$

$$x_1(k+1) = [A_{11} + A_{12}A_{21}A_{11}^{-1}]x_1(k) + A_{12}[A_{22} - A_{21}A_{11}^{-1}A_{12}]x_2(k-1)$$

$$+ A_{12}[B_2 - A_{21}A_{11}^{-1}B_1]u(k-1) + B_1u(k)$$

By successive substitution of

$$x_2(k-j) = [A_{22} - A_{21}A_{11}^{-1}A_{12}]x_2(k-(j+1)) + [B_2 - A_{21}A_{11}^{-1}B_1]u(k-(j+1))$$

$$+ A_{21}A_{11}^{-1}x_1(k-j) \qquad (3.30)$$

we again are "reflectintng back" or shifting μ terms one unit forward and thus approximating a "future" system with less system order. As a result we get

$$x_1(k+1) = [A_{11} + A_{12}A_{21}A_{11}^{-1}]x_1(k) + B_1u(k)$$

$$+A_{12}[A_{22}-A_{21}A_{11}^{-1}A_{12}]A_{21}A_{11}^{-1}x_1(k-1)$$

$$+A_{12}[B_2-A_{21}A_{11}^{-1}B_1]u(k-1)$$

$$+A_{12}[A_{22}-A_{21}A_{11}^{-1}A_{12}]^2A_{21}A_{11}^{-1}x_1(k-2)$$

$$+A_{12}[A_{22}-A_{21}A_{11}^{-1}A_{12}]^1[B_2-A_{21}A_{11}^{-1}B_1]u(k-2)$$

$$+.....+$$

$$+A_{12}[A_{22}-A_{21}A_{11}^{-1}A_{12}]^{N-1}A_{21}A_{11}^{-1}x_1(k-(N-1)) \qquad (3.31)$$

$$+A_{12}[A_{22}-A_{21}A_{11}^{-1}A_{12}]^{N-2}A_{21}A_{11}^{-1}u(k-(N-1))$$

$$+A_{12}[A_{22}-A_{21}A_{11}^{-1}A_{12}]^{N}x_2(k-N)$$

$$+A_{12}[A_{22}-A_{21}A_{11}^{-1}A_{12}]^{N-1}[B_2-A_{21}A_{11}^{-1}B_1]u(k-N)$$

Now, by explicity entering μ, μ^j, $\mu^{(1-j)}$

$$x_1(k+1) = [A_{11}+\mu\hat{A}_{12}\hat{A}_{21}A_{11}^{-1}]x_1(k) + B_1u(k)$$

$$+\mu^2\hat{A}_{12}[\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12}]\hat{A}_{21}A_{11}^{-1}x_1(k-1)$$

$$+\mu^{(1)}\hat{A}_{12}[\hat{B}_2-\hat{A}_{21}A_{11}^{-1}B_1]u(k-1) \qquad (3.32)$$

$$+\mu^3\hat{A}_{12}[\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12}]^2\hat{A}_{21}A_{11}^{-1}x_1(k-2)$$

$$+\mu^{(2)}\hat{A}_{12}[\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12}]^1[\hat{B}_2-\hat{A}_{21}A_{11}^{-1}B_1]u(k-2)$$

$$+.....+$$

$$+\mu^{N}\hat{A}_{12}[\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12}]^{N-1}\hat{A}_{21}A_{11}^{-1}x_1(k-(N-1))$$

$$+\mu^{N-1}\hat{A}_{12}[\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12}]^{(N-2)}[\hat{B}_2-\hat{A}_{21}A_{11}^{-1}B_1]u(k-(N-1))$$

$$+\mu^{N+1-j}\hat{A}_{12}[\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12}]^{N}x_2(k-N)$$

$$+\mu^{N}\hat{A}_{12}[\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12}]^{(N-1)}[\hat{B}_2-\hat{A}_{21}A_{11}^{-1}B_1]u(k-N)$$

With this expansion, we can now define some low order models for the $x_1(k+1)$ subsystem. To $0(\mu)$

To $0(\mu)$

$$x_1(k+1) = A_{11}x_1(k) + B_1u(k) \tag{3.33}$$

To $0(\mu^2)$

$$x_1(k+1) = [A_{11} + \mu\hat{A}_{12}\hat{A}_{21}A_{11}^{-1}]x_1(k) + B_1u(k)$$

$$+\mu\hat{A}_{12}[\hat{B}_2-\hat{A}_{21}A_{11}^{-1}B_1]u(k-1) \tag{3.34}$$

In matrix form

$$\begin{bmatrix} x_1(k+1) \\ \hline z(k+1) \end{bmatrix} = \left[\begin{array}{c|c} (A_{11}+\mu\hat{A}_{12}\hat{A}_{21}A_{11}^{-1}) & \mu\hat{A}_{12}(\hat{B}_2-\hat{A}_{21}A_{11}^{-1}B_1) \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} x_1(k) \\ \\ z(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \\ I \end{bmatrix} u(k) \tag{3.35}$$

This model possesses some interesting properties some of which will be made apparent in the remaining chapters.

4. REDUCED ORDER CONTROL PROBLEMS

4.1. Introduction

In the previous two chapters we have shown the existence of the slow-fast phenomena for any system fitting our general model. Up to now, our reduced order modeling has been applied only to system simulation. In this chapter we begin the problem of designing control laws based on our reduced order subsystems and applying these control laws to our higher order models.

When designing stabilizing control laws for linear shift-invariant discrete-time systems, perhaps the most frequently used methods are the optimal linear regulator design and pole placement. This chapter will deal with reduced order pole placement procedures. Reduced order regulator designs is another research topic in itself and will not be considered here.

When we consider any reduced or approximated procedure, the two important factors are:

1) computations involved

2) accuracy when implemented on high order model.

The previous chapters helped to show us the computations involved in attaining some reduced order models. Since we have assumed stable fast modes, we took advantage of the fast "transient" to develop "steady state" models of our original system. These models helped to verify our slow-fast block diagonalization transformations and to introduce the "boundary layer" phenomena of the $x_2(k)$ subsystem. In this chapter, we take this idea one step further by now incorporating a feedback control variable into the design. Since we will be dealing with "slow" and "fast" eigenvalue placement

we will, for the most part, find it easier working with the transformed state vectors $x_s(k)$ and $x_f(k)$ rather than with our original state vectors $x_1(k)$ and $x_2(k)$.

In the next section we will give some basic differences between singular perturbations in continuous systems and reduced order modeling in two-time-scale discrete systems.

4.2. Correspondence Between Discrete Two-Time-Scale Systems and Continuous Singular Perturbations

Our dual slow-fast transformations, in general form, are given as

S:
$$\begin{aligned} x_1(k) &= (I-P^sQ^s)x_s(k) - P^s x_f(k) \\ x_2(k) &= Q^s x_s(k) + x_f(k) \end{aligned} \tag{4.1}$$

F:
$$\begin{aligned} x_1(k) &= x_s(k) + Q^s x_f(k) \\ x_2(k) &= -P^f x_s(k) + (I-P^fQ^f)x_f(k). \end{aligned} \tag{4.2}$$

Each of our original subsystem variables is a function of our transformed slow and fast variables.

In many instances, when a design model is being formulated, the fast modes are neglected. It is the purpose of this chapter to examine reduced order approximations to our original systems when the fast modes are neglected or when their control design is to be carried out separately.

This has been well documented for the continuous case [4,5] where the perturbation made in the original system is singular, and the order reduction achieved is from $N \rightarrow N_s$.

However, in trying to adapt these ideas to the discrete case, a fundamental concept between two-time-scale discrete and continuous systems must be understood.

In continuous systems, the perturbation allowing the real part of the fast eigenvalues to go to $-\infty$ is the same perturbation involved in stating that the $\dot{x}_2(t)$ subsystem goes to equilibrium instantaneously.

Using the block diagonalization transformation defined in [12] on a continuous two-time-scale system, we get

$$\begin{bmatrix} \dot{x}_s \\ \hline \dot{x}_f \end{bmatrix} = \begin{bmatrix} A_o & \vdots & 0 \\ \hdashline 0 & \vdots & A_{22} \end{bmatrix} \begin{bmatrix} x_s \\ \hline x_f \end{bmatrix} + 0(\mu).$$

Neglecting the fast modes is equivalent to letting the

$$\mathrm{Re}\{\lambda[A_{22}]\} \to -\infty \text{ or } x_f(t) = 0 \quad \forall t > 0.$$

When our two-time-scale system is put in singularly perturbed form, it looks like

$$\begin{bmatrix} \dot{x}_1 \\ \hline \mu\dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & \vdots & A_{12} \\ \hdashline A_{21} & \vdots & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix}. \tag{4.4}$$

By letting $\mu \to 0$ and assuming $\mathrm{Re}\{\lambda[A_{22}]\} < 0$, we are again assuming that $\mathrm{Re}\{\lambda[A_{22}]\} \to -\infty$ either way, the perturbation is consistent. In the discrete case, the fast modes are not totally contained in the A_{22} submatrix or any other submatrix to any order. Therefore, letting $\lambda[A_f] \to 0$ and $[x_2(k+1)-x_2(k)] \to 0$ are not equivalent.

The problem is that for the general structure we have assumed, the fast submatrix to $0(\mu^2)$ is given by:

$$A_f = A_{22} - A_{21}A_{11}^{-1}A_{12} + 0(\mu^2)$$

and it is therefore very difficult to isolate the fast modes in the original system. This is possible only through a canonical form.

Another more serious problem of discrete two-time-scale systems is that of "relative order" of system parameters.

What this means is that the μ-parameters present in the discrete model also correspond to the order of the fast eigenvalues. So, for small μ, an independent, reduced order "slow" design will give good results, however, since our fast λ's are all ready $0(\mu)$, an accurate reduced order fast design is almost impossible without including many series terms. As a result, we will show that by iterating the P_k^f matrix we will be able to solve our fast problem to a desired accuracy.

In the next section we comment on controllability aspects of our reduced order models.

4.3. Controllability Aspects of Reduced Order Models

In this section, we discuss some controllability bounds on our two-time-scale system such that our reduced order design models may be used.

Given a system in our general form

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & \mu^{(1-j)}\hat{A}_{12} \\ \hline \mu^{j}\hat{A}_{21} & \mu\hat{A}_{22} \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_1 \end{bmatrix} u(k). \tag{4.5}$$

Satisfying the usual requirements, we have shown that for μ sufficiently small, there exists a nonsingular transformation T, such that

$$\begin{bmatrix} x_1(k) \\ - - - \\ x_2(k) \end{bmatrix} = T^{-1} \begin{bmatrix} x_s(k) \\ - - - \\ x_f(k) \end{bmatrix} \quad \text{where } \|T^{-1}\| \triangleq 0(1)$$

$$\begin{bmatrix} x_s(k+1) \\ - - - - \\ x_f(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_s(\mu) & 0 \\ - - - & - - - \\ 0 & A_f(\mu) \end{array}\right] \begin{bmatrix} x_s(k) \\ - - - \\ x_f(k) \end{bmatrix} + \begin{bmatrix} B_s(\mu) \\ - - - \\ B_f(\mu) \end{bmatrix} u(k). \tag{4.6}$$

If T is, for example, the "F" transformation,

$$\begin{aligned} A_s &= A_{11} - \mu^{(1-j)} \hat{A}_{12} P_N \\ A_f &= \mu \hat{A}_{22} + \mu^{(1-j)} P_N \hat{A}_{12} \\ B_s &= (I - Q_N P_N) B_1 - Q_N B_2 \\ B_f &= P_N B_1 + B_2 \end{aligned} \tag{4.7}$$

where

$$P_{k+1} = [\mu^{(1-j)} P_k \hat{A}_{12} P_k + \mu \hat{A}_{22} P_k - \mu^j \hat{A}_{21}] A_{11}^{-1} \tag{4.8}$$

$$P_o = -\mu^j \hat{A}_{21} A_{11}^{-1}$$

$$Q_{k+1} = A_{11}^{-1} [\mu^{(1-j)} P_N Q_k + \mu Q_k \hat{A}_{22} + \mu^j Q_k P_N \hat{A}_{12} - \mu^{(1-j)} \hat{A}_{12}] \tag{4.9}$$

$$Q_o = -\mu^{(1-j)} A_{11}^{-1} \hat{A}_{12}.$$

As the P and Q matrix recursions approach equilibrium, since $\lambda(A_s(\mu))$ and $\lambda(A_f(\mu))$ are sufficiently far apart, controllability of our original system pair (A,B) guarantees controllability of our subsystem pairs

$$(A_s(\mu), B_s(\mu)), \quad (A_f(\mu), B_f(\mu)).$$

However, in real time applications, computation time requirements will make exact equilibrium values of P and Q unreachable in general. Thus,

$$\begin{aligned} P &= P_N + 0(\mu^{N+j}) \\ Q &= Q_N + 0(\mu^{N+1-j}) \end{aligned}$$

resulting in

$$A_s(\mu) = \hat{A}_s + 0(\mu^N)$$
$$A_f(\mu) = \hat{A}_f + 0(\mu^N)$$
$$B_s(\mu) = \hat{B}_s + 0(\mu^{N+1-j})$$
$$B_f(\mu) = \hat{B}_f + 0(\mu^{N+j}).$$

When this is the case, bounds on the controllability of our reduced order models are best defined in terms of "weak" and "strong" controllability conditions of perturbed systems defined by Chow [17]. For a given finite number of iterations N of the P and Q matrices, it is necessary, for separate design of slow and fast subsystems that our original system be strongly controllable in μ^N (i.e. if we truncate (4.8) and (4.9) after N iterations, (4.5) and (4.6) will still span the same controllable subspace). This is analogous to the condition defined by Chow [18] for continuous singularly perturbed systems.

Expressing (4.6) as

$$\begin{bmatrix} x_s(k+1) \\ x_f(k+1) \end{bmatrix} = \begin{bmatrix} \hat{A}_s + 0(\mu^N) & 0 \\ 0 & \hat{A}_f + 0(\mu^N) \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} + \begin{bmatrix} \hat{B}_s + 0(\mu^{N+1-j}) \\ \hat{B}_f + 0(\mu^{N+j}) \end{bmatrix} u(k).$$

We can readily apply Theorem 1 of [12] to establish bounds on μ^N to verify the strong controllability of our original system.

Theorem 7: If A_{11}^{-1} exists and if

$$\text{rank}[\hat{B}_s, \hat{A}_s\hat{B}_s, \ldots, \hat{A}_s^{N-1}\hat{B}_s] = N_s$$
$$\text{rank}[\hat{B}_f, \hat{A}_f\hat{B}_f, \ldots, \hat{A}_f^{N-1}\hat{B}_f] = N_f$$

then there exists $\mu^* \ni$ (4.10) is controllable $\forall \mu \in (0, \mu^*]$.

In other words, (4.5) is strongly controllable in that as $\mu^N \to 0$, our fast and slow subsystems maintain their controllability.

By checking the conditions of Theorem 7, we can, in theory, establish a lower bound on the number of iterations of P and Q.

4.4. $0(\mu)$, $0(\mu^2)$ Control Approximations

The remainder of this chapter deals with finding control laws based on our reduced order models and observing their accuracy when implemented on the higher order system.

Control law design of continuous singularly perturbed systems has been well documented [4,5,7]. The reduced order control law designs for eigenvalue placement [5] and the optimal linear regulator [4,7] achieve $0(\mu)$ accuracy with respect to exact feedback gains. This accuracy is usually very acceptable since the open loop fast eigenvalues have real parts that are $0(1/\mu)$ and the slow eigenvalues have real parts that are $0(1)$. Also, when expressed in block diagonal slow-fast form [12], the continuous singularly perturbed system has the nice property that $\|B_f\| \triangleq 0(1/\mu)$ while $\|B_s\| \triangleq 0(1)$. This enables you, to $0(\mu)$ accuracy, to decompose the control into independent slow and fast as well as the homogeneous system dynamics. Thus, the three orders of magnitude present in continuous singularly perturbed systems ($0(\mu)$, $0(1)$, $0(1/\mu)$), make independent $0(\mu)$ accurate approximations and decisions very acceptable and appealing, to large scale systems possessing this property.

In discrete two-time-scale systems such as we have defined, the "relative order" of parameters does not permit such $0(\mu)$ approximations to

be acceptable. The $\|A_f\| \triangleq 0(\mu)$ so any $0(\mu)$ approximation would only be meaningful if we just neglect the fast modes and design a reduced order slow control ($\|A_s\| \triangleq 0(1)$). Likewise, depending on μ, $0(\mu^2)$ approximations could be unacceptable for both fast and slow subsystems. Also, in the model we have used, no assumptions have been made on the relative order between $\|B_1\|$ and $\|B_2\|$. This enables us to encompass a greater class of discrete-time systems which have open-loop two-time-scale properties. Granted, by making order assumptions on $\|B_1\|$, $\|B_2\|$, independent control decomposition may be possible, as will be the case for sampled-data singularly perturbed systems. However, for now, we are considering purely discrete-time systems, so the proposed generality will remain.

As a result of these problems concerning two-time-scale discrete systems, we divide our reduced order control law design into two parts. In this section, $0(\mu)$ and $0(\mu^2)$ approximations are considered. In other words, 0 or 1 iteration is to be used in our successive approximation scheme to keep computations to a minimum. Thus, we achieve a reduced order slow representation of our high order system accurate to $0(\mu)$ or $0(\mu^2)$ for the purpose of <u>slow</u> control law design. Of course, the price we pay is insufficient accuracy to use a reduced order fast model to generate a <u>fast</u> control law. The only requirement we can pose on such a design, is to place order N_s slow mode where we want, and keep original N_f fast modes as unaltered as possible.

In the next two sections, we take advantage of our simple successive approximation matrix recursions to obtain fast and slow reduced order subsystems to $0(\mu^N)$ accuracy, where N will be the number of iterations.

This enables us to generate highly accurate slow and fast models for the purpose of separate fast and slow control law designs. The price you pay is, of course, more computations than the $0(\mu)$ approach, however, we can now specify both fast and/or slow modes to a desired accuracy, using much less computation than a high-order pole placement algorithm applied to such an ill-conditioned system. Also, if the feedback gains are to be altered in some adaptive scheme, this reduced order method offers even further computational simplicity.

We now proceed with our $0(\mu)$, $0(\mu^2)$ designs. Before considering $0(\mu)$ approximations to our general model, let us consider one of our limiting cases, say "V." Here, the perturbation $\mu \to 0$ has meaning in both the actual and transformed state space. Given the system

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & \mu\hat{A}_{12} \\ \hline A_{21} & \mu\hat{A}_{22} \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u(k) \tag{4.11}$$

where i. $\underline{x}_1(k) \in R^{N_s}$, $x_2(k) \in R^{N_f}$, $\underline{u}(k) \in R^m$

ii. A_{ij}, B_j are real constant-coefficient matrices

iii. μ satisfies Corollary 5.

Then we showed that there exists a nonsingular transformation (2.28) such that (4.11) may be transformed into

$$\begin{bmatrix} x_s(k+1) \\ \hline x_f(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11}-\mu\hat{A}_{12}L & 0 \\ \hline 0 & \mu\hat{A}_{22}+\mu\hat{A}_{12} \end{array}\right] \begin{bmatrix} x_s(k) \\ \hline x_f(k) \end{bmatrix} + \begin{bmatrix} (I-ML)B_1-MB_2 \\ \hline LB_1+B_2 \end{bmatrix} u(k) \tag{4.12}$$

where L and M are equilibrium solutions of

$$L_{k+1} = [L_k A_{12} L_k + A_{22} L_k - A_{21}] A_{11}^{-1} \quad L_o = -A_{21}A_{11}^{-1}$$

$$M_{k+1} = A_{11}^{-1}[A_{12}LM_k + M_kA_{22} + M_kLA_{12} - A_{12}]$$

$$A_{12} = \mu\hat{A}_{12}, \quad A_{22} = \mu\hat{A}_{22} \qquad M_o = -A_{11}^{-1}A_{12}$$

and

$A_s \triangleq A_{11} - A_{12}L$ has N_s slow eigenvalues

$A_f \triangleq A_{22} + LA_{12}$ has N_f fast eigenvalues.

For μ sufficiently small, a designer may simply ignore μ, which is equivalent to assuming

$$|\lambda_i(A_f)| = 0 \qquad i = 1,2,\ldots,N_f$$

or, all our fast modes are assumed to have a deadbeat response. When this perturbation is made on (4.11)

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & 0 \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u(k). \tag{4.13}$$

The designer, however, uses his $0(\mu)$ slow model and chooses some feedback gain G to place the N_s slow eigenvalues of the reduced order system

$$x_1(k+1) = A_{11}x_1(k) + B_1u(k) \tag{4.14}$$

where $u(k) = Gx_1(k)$ providing the pair (A_{11},B_1) is controllable. It is interesting to note that (4.14) will result if $\mu = 0$ in (3.32). This further verifies that to $0(\mu)$, $x_1(k)$ will determine the slow dynamics of the actual system.

With feedback included, the closed loop system is now

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} (A_{11}+B_1G) & 0 \\ \hline (A_{21}+B_2G) & 0 \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} \tag{4.15}$$

which preserves the deadbeat approximation of the fast modes in accordance with our model.

How accurate is this reduced order design?

i. How close to your desired "slow" eigenvalues are your actual eigenvalues?

ii. How much have your fast eigenvalues been proved as a result of this feedback?

Lemma 8: Given system (4.5), if (A_{11},B_1) is a controllable pair, then by choosing N_s slow locations for eigenvalues using (4.11)

$$\lambda_{slow}(new) = \lambda_{slow}(desired) + 0(\mu) \tag{4.13a}$$

$$\lambda_{fast}(new) = \lambda_{fast}(old) + 0(\mu). \tag{4.13b}$$

Proof: Both (4.13a) and (4.13b) are are obvious consequences of our original transformation. Our original system (4.11) with this reduced feedback looks like

$$\begin{bmatrix} x_1(k+1) \\ - - - \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11}+B_1G & \vdots & \mu\hat{A}_{12} \\ - - - - & \vdots & - - \\ A_{21}+B_2G & \vdots & \mu\hat{A}_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ - - - \\ x_2(k) \end{bmatrix}. \tag{4.14}$$

The slow eigenvalues of this system will be the eigenvalues of

$$(A_{11}+B_1G) + \mu(A_{21}+B_2G)(\hat{A}_{12})(A_{11}+B_1G)^{-1} + 0(\mu^2)$$

verifying (4.13a). The fast eigenvalues are the eigenvalues of

$$\mu(\hat{A}_{22}-(A_{21}+B_2G)(A_{11}+B_1G)^{-1}(\hat{A}_{12})) + 0(\mu^2) \tag{4.15}$$

as compared with the old eigenvalues of

$$\mu(\hat{A}_{22}-A_{21}A_{11}^{-1}\hat{A}_{12}) + 0(\mu^2)$$

which trivially verifies (4.13b), but more importantly, establishes a bound on the magnitude of the fast closed loop eigenvalues.

Example: To verify these accuracy results, let us try varying μ in the following system after we generate a control for the low order model.

Given

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \hline x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 1 & \mu\begin{pmatrix} 1 & 1 \\ 0 & .1 \end{pmatrix} \\ \hline 0 & 1 & \\ 1 & -1 & \mu\begin{pmatrix} .1 & .5 \\ 0 & 1 \end{pmatrix} \\ .2 & 0 & \end{array}\right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \hline x_3(k) \\ x_4(k) \end{bmatrix} + \left[\begin{array}{c} 1 \\ 1 \\ \hline 1 \\ 1 \end{array}\right] u(k)$$

let us first find a feedback G such that

$$u(k) = Gx_1(k)$$

and the eigenvalues of $(A_{11} + B_1 G) \triangleq .96 \pm .1j$ using LINSYS [16] to do this we get

$$G = [-.0116, \quad -.0684].$$

Now, applying this feedback to our original system we will show the actual eigenvalues for varying values of μ.

μ	Slow Eigenvalues	Fast Eigenvalues
0	$.96 \pm .1j$	0, 0
.01	$.96568 \pm .09582j$	.009699, -.001055
.05	$.9874 \pm .0741j$	.04321, -.04321
.1	1.0206, 1.0057	.08617, -.08479
.4	1.37, .9248	.3378, -.27269

If the $0(\mu)$ design is unsatisfactory, the logical approach might be to take another term in the series of (3.32) and thus use 3.35) given here again for convenience.

$$\begin{bmatrix} x_1(k+1) \\ \hline z(k+1) \end{bmatrix} = \left[\begin{array}{c|c} (A_{11}+\mu\hat{A}_{12}A_{21}A_{11}^{-1}) & \mu\hat{A}_{12}(B_2-A_{21}A_{11}^{-1}B_1) \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline I \end{bmatrix} u(k). \quad (4.17)$$

By using this model, we should be able to model the $x_1(k)$ states to $0(\mu^2)$ accuracy. Thus, the question arises; to what accuracy can this model be used to place the slow eigenvalues?

If we truncate our transformed slow variable $x_s(k+1)$ to $0(\mu^2)$ accuracy, we get

$$\underline{x}_s(k+1) = [A_{11}+\mu\hat{A}_{12}A_{21}A_{11}^{-1}]x_s(k) + [B_1+\mu A_{11}^{-1}\hat{A}_{12}(B_2-A_{21}A_{11}^{-1}B_1)]u(k). \quad (4.18)$$

We immediately notice the absence of the delayed control term. To see a reason for this, let us look at the components of $x_s(k)$ for structure "V" using the "F" transformation

$$x_s(k) = (I-ML)x_1(k) - Mx_2(k).$$

We have already shown that $\|M\| \triangleq 0(\mu)$. Therefore

$$x_s(k) = x_1(k) + 0(\mu)$$

$$x_s(k) \neq x_1(k) + 0(\mu^2).$$

Thus, while model (4.17) accurately defines the $x_1(k)$ states to $0(\mu^2)$, we see that past $0(\mu)$ $x_s(k) \neq x_1(k)$ in the nonhomogeneous case. So, for eigenvalue placement of slow modes, it is necessary to consider (4.18). So, to place our slow eigenvalues to $0(\mu^2)$ accuracy consider the feedback

$$u_s(k) = G_s x_s(k)$$

when applied to

$$x_s(k+1) = \underbrace{[A_{11}+\mu\hat{A}_{12}A_{21}A_{11}^{-1}]}_{A_s} x_s(k) + \underbrace{[(I-\mu A_{11}^{-1}\hat{A}_{12}A_{21}A_{11}^{-1}B_1 + \mu A_{11}^{-1}\hat{A}_{12}B_2]}_{B_s} u(k) \tag{4.20}$$

and design G_s to place N_s slow eigenvalues of

$$(A_s + B_s G_s).$$

Then we may form our control

$$\begin{aligned} u_s(k) &= G_s x_s(k) \\ &= G_s[(I-\mu A_{11}^{-1}\hat{A}_{12}A_{21}A_{11}^{-1})x_1(k) + \mu A_{11}^{-1}\hat{A}_{12}x_2(k)] \\ &= G_s(I-\mu A_{11}^{-1}\hat{A}_{12}A_{21}A_{11}^{-1})x_1(k) + \mu G_s A_{11}^{-1}\hat{A}_{12}x_2(k) \end{aligned}$$

and place our slow eigenvalues to within $0(\mu^2)$ of their desired values while changing our fast eigenvalues by $0(\mu^2)$. Now, what if the discrete-time system does not fit structure "V," but does fit the general model? When this is the case, we will show how the same reduced order model result. However, the perturbations must be carried out in the transformed state domain.

If the transformed system (2.43) is truncated after one iteration

$$\begin{bmatrix} x_s(k+1) \\ \hline x_f(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11}+\mu^{(1-j)}\hat{A}_{12}\mu^{j}\hat{A}_{21}A_{11}^{-1} & 0 \\ \hline 0 & \mu\hat{A}_{22}-\mu^{j}\hat{A}_{21}A_{11}^{-1}\mu^{(1-j)}\hat{A}_{12} \end{array}\right] \begin{bmatrix} x_s(k) \\ \hline x_f(k) \end{bmatrix}$$

$$+ \left[\begin{array}{c} (I-A_{11}^{-1}\mu^{(1-j)}\hat{A}_{12}\mu^{j}\hat{A}_{21}A_{11}^{-1})B_1 + A_{11}^{-1}\mu^{(1-j)}\hat{A}_{12}B_2 \\ \hline -\mu^{j}\hat{A}_{21}A_{11}^{-1}B_1 + B_2 \end{array}\right] u(k) \tag{4.22}$$

and since μ is a positive scalar,

$$\begin{bmatrix} x_s(k+1) \\ \hline x_f(k+1) \end{bmatrix} = \begin{bmatrix} A_{11}+\mu\hat{A}_{12}\hat{A}_{21}A_{11}^{-1} & 0 \\ \hline 0 & \mu(\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12}) \end{bmatrix} \begin{bmatrix} x_s(k) \\ \hline x_f(k) \end{bmatrix}$$

$$+ \begin{bmatrix} (I-\mu A_{11}^{-1}\hat{A}_{12}\hat{A}_{21}A_{11}^{-1})B_1 + \mu A_{11}^{-1}\hat{A}_{12}\hat{B}_2 \\ \underbrace{(B_2-\mu^j\hat{A}_{21}A_{11}^{-1}B_1)}_{w^*} \end{bmatrix} u(k) \qquad (4.23)$$

where $\|w^*\| = f(j)$.

Now, if $\mu \to 0$

$$\begin{bmatrix} x_s(k+1) \\ \hline x_f(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ \hline 0 & 0 \end{bmatrix} \begin{bmatrix} x_s(k) \\ \hline x_f(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline w^* \end{bmatrix} u(k). \qquad (4.24)$$

Again, the dynamics of the system will be governed by the pair (A_{11}, B_1), and your reduced order model becomes

$$x_s(k+1) = A_{11}x_s(k) + B_1u_s(k)$$

and $u_s(k) = Gx_s(k)$ can be used to place N_s slow modes at desired slow locations. However, when implementing the control

$$\begin{aligned} u_s(k) &= G_s x_s(k) \\ &= G_s[(I-\mu A_{11}^{-1}\hat{A}_{12}\hat{A}_{21}A_{11}^{-1})x_1(k) + \mu^{(1-j)}A_{11}^{-1}\hat{A}_{12}x_2(k)]. \end{aligned} \qquad (4.25)$$

So as $\mu \to 0$, $x_1(k) \neq x_s(k)$ in general. Thus, your actual feedback will be of the form

$$u(k) = G_s x_1(k) + G_s\mu^{(1-j)}A_{11}^{-1}\hat{A}_{12}x_2(k).$$

The $x_2(k)$ term may or may not be neglected depending on the value of j.

$0(\mu^2)$ and higher accuracy reduced order models are resolved in the same manner.

Example $0(\mu^2)$ design: Consider the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \mu\begin{pmatrix} 1 & 1 \\ 0 & .1 \end{pmatrix} \\ 0 & 1 & \\ 1 & -1 & \mu\begin{pmatrix} .1 & .5 \\ 0 & 1 \end{pmatrix} \\ .2 & 0 & \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u(k).$$

In a previous example, for $\mu = .1, .4$, our $0(\mu)$ design proved unstable. Now, for these same values of μ, let us try an $0(\mu^2)$ design. $\mu = .1$

$$x_s(k+1) = \begin{bmatrix} 1.12 & .78 \\ .0002 & .9998 \end{bmatrix} x_s(k) + \begin{bmatrix} 1.299 \\ 1.001 \end{bmatrix} u(k).$$

For λ's of $.96 \pm .1j$

$$u_s(k) = (-.038122, -.15013)x_s(k)$$

$$= (-.033525, -.15854)x_1(k) + (-.0038122, -0039242)x_2(k)$$

which gives us closed loop eigenvalues of

$$.95534 \pm .10174j, -.086308, .085836$$

opposed to open loop eigenvalues of

$$1.1238, .98549, -.09298, .084659.$$

For $\mu = .4$

$$x_s(k+1) = \begin{bmatrix} 1.48 & .12 \\ .008 & .992 \end{bmatrix} x_s(k) + \begin{bmatrix} 2.16 \\ 1.04 \end{bmatrix} u(k)$$

$$u_s(k) = (-.23814 \quad -.036177)x_s(k)$$

$$= (-.12545 \quad -.24412)x_1(k) + (-.095256 \quad -.087177)x_2(k)$$

which gives closed-loop eigenvalues of

$$.97222 \pm .2019j, \quad -.3854, \quad +.32901$$

opposed to open-loop eigenvalues of

$$1.4, \quad .98689, \quad .3317, \quad -.2786.$$

4.5. High-Accuracy Reduced Order Modeling for Eigenvalue Placement

In the previous section, we have shown how reduced order models can be used to place N_s "slow" eigenvalues to $0(\mu)$ and $0(\mu^2)$ accuracy. However, what if for any one of a number of reasons, you would want to be able to place the N_f fast eigenvalues at N_f fast stable locations. By approaching the problem from our limiting cases "V" and "H" first, we hope to point out the advantages and disadvantages that each transformation possesses before applying both transformations to the control design of our general model. Also, what if our control algorithm involved an adaptive/identification procedure such that gains had to be updated in some multi-processing environment. It would then be desirable to have an algorithm of low order computationally, but very accurate that could be easily updated if necessary.

In this section we will propose two high-accuracy reduced-order modeling algorithms for the purpose of eigenvalue placement. Again, the algorithms will be dual to one another. Also, various structural properties of the system will enable us to improve the algorithms efficiency.

We will first consider an algorithm for structure "V." A dual procedure for "H" will be outlined in much less detail. Finally, an algorithm for our general model will then be given in our next section.

Part I: Structure "V"

Given

$$\begin{bmatrix} x_1(k+1) \\ - - - \\ x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & \mu\hat{A}_{12} \\ - - & - - \\ A_{21} & \mu\hat{A}_{22} \end{array}\right] \begin{bmatrix} x_1(k) \\ - - - \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ - - \\ B_2 \end{bmatrix} u(k)$$

where i. $\underline{x}_1(k) \in R^{N_s}$, $\underline{x}_2(k) \in R^{N_f}$

ii. A_{ij}, B_j, $i,j=1,2$ are real constant coefficient matrices

iii. μ is a small positive parameter such that

$$0 \leq \mu < \frac{1}{3(\|A_{11}^{-1}\|(\|A_o\|+\|L_o\|\|\hat{A}_{12}\|)} \qquad A_o = A_{22} + L_o\hat{A}_{12}.$$

Iterate the following

$$L_{k+1} = (\mu\hat{A}_{22}L_k + \mu L_k\hat{A}_{12}L_k - A_{21})A_{11}^{-1} \qquad L_o = A_{21}A_{11}^{-1}. \tag{4.26}$$

The accuracy you desire will depend on the number of iterations. Letting "L" be the equilibrium value, the kth iteration will give the following accuracy

$$L = L_k + O(\mu^{k+1}).$$

Now, place N_f fast modes using the fast subsystem

$$x_f(k+1) = \underbrace{\mu[\hat{A}_{22} + L\hat{A}_{12}]}_{A_f} x_f(k) + \underbrace{[LB_1 + B_2]}_{B_f} u(k) \tag{4.27}$$

$$\begin{aligned} u_f(k) &= G_f x_f(k) \\ &= G_f[Lx_1(k) + x_2(k)] \\ &= G_f L x_1(k) + G_f x_2(k). \end{aligned} \tag{4.28}$$

It is important now to look at this partial feedback in both our transformed system and original system. In the transformed system,

$$\begin{bmatrix} x_s(k+1) \\ \hline x_f(k+1) \end{bmatrix} = \begin{bmatrix} A_s & \vdots & B_s G_f \\ \hline 0 & \vdots & A_f + B_f G_f \end{bmatrix} \begin{bmatrix} x_s(k) \\ \hline x_f(k) \end{bmatrix} + \begin{bmatrix} B_s \\ \hline B_f \end{bmatrix} u(k). \tag{4.29}$$

Notice, if we were to reblockdiagonalize this system (i.e. $x_f^*(k) = x_f(k) - M^* x_s(k)$), our block diagonal terms would not change. Thus, our slow A_s submatrix would be preserved. Plus, our fast eigenvalues would be defined by our designed fast subsystem $(A_f + B_f G_f)$. Thus, the only accuracy limitations will be on the number of iterations used in (4.26) to generate our reduced order models when (4.28) is fed back to our original system,

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} + B_1 G_f L & \vdots & \mu \hat{A}_{12} + B_1 G_f \\ \hline A_{21} + B_2 G_f L & \vdots & \mu \hat{A}_{22} + B_2 G_f \end{bmatrix} \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u(k). \tag{4.30}$$

Let us look at

$$\lambda \left(\underbrace{(\mu(\hat{A}_{22} + \mu L \hat{A}_{12}) + (LB_1 + B_2) G_f)}_{AF} \right) .$$

We have designed G_f to place the N_f fast modes in N_f fast desired (stable) locations. Therefore, AF will define our fast subsystem when in block diagonal form. Thus,

$$\|AF\| \triangleq O(\mu^*) \quad \text{where } \mu^* \neq \mu .$$

This defines our fast mode to be within some radius of the origin in the z-plane. If we let

$$\mu_M = \max(\mu, \mu^*) .$$

Now our partially closed loop system may be written as:

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11}+\mu_M B_1^* G_f^* L & \mu_M(A_{12}^*+B_1^* G_f^*) \\ \hline A_{21}+\mu_M B_2^* G_f^* L & \mu_M(A_{22}^*+B_2^* G_f^*) \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u(k)$$

$$= \left[\begin{array}{c|c} F_{11} & \mu_M \hat{F}_{12} \\ \hline F_{21} & \mu_M \hat{F}_{22} \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u(k) \qquad (4.31)$$

where

$$A_{12}^* = \frac{\hat{A}_{12}\mu}{\mu_M}, \quad B_1^* G_f^* = \frac{B_1 G_f}{\mu_M}, \quad B_2^* G_f^* = \frac{B_2 G_f}{\mu_M}.$$

Now, if

$$0 \le \mu_M < \frac{1}{3\|F_{11}^{-1}\|\,(\|F_o\|+\|L_o^*\|\|\hat{F}_{12}\|)}. \qquad (4.32)$$

Then the following iteration will converge

$$L_{k+1}^* = [\mu_M \hat{F}_{22} L_k^* + \mu_M L_k^* \hat{F}_{12} L_k^* - F_{21}] F_{11}^{-1} \qquad (4.33)$$

$$L_o = -F_{21} F_{11}^{-1}.$$

We now state a lemma which relates L to L^* and could save computations.

Lemma 9: If $N_f \ge N_s$ and if the matrix $(\mu A_{12}+B_1 G_f)$ is of full rank, the L matrix is invariant to the partial feedback

$$u_f(k) = G_f L x_1(k) + G_f x_2(k)$$

and therefore $L = L^*$.

Proof: As we have already shown, the partial feedback will not affect the structure of the homogeneous slow submatrix. Therefore

$$(F_{11}-\mu_M \hat{F}_{12} L^*) = (A_{11}-\mu \hat{A}_{12} L)$$

$$(A_{11}+\mu_M B_1^* G_f^* L) - \mu_M (A_{12}^*+B_1^* G_f^*) L^* = A_{11} - \mu \hat{A}_{12} L$$

$$A_{11} + \mu_M B_1^* G_f^* L - \mu_M A_{12}^* L^* - \mu_M B_1^* G_f^* L^* = A_{11} - \mu \hat{A}_{12} L.$$

However, since $\mu_M A_{12}^* = \mu \hat{A}_{12}$, $\mu_M B_1^* G_f^* = B_1 G_f$

$$B_1 G_f L - \mu\hat{A}_{12} L^* - B_1 G_f L^* = -\mu\hat{A}_{12} L$$

$$B_1 G_f (L-L^*) + \mu\hat{A}_{12} (L-L^*) = 0$$

$$\underbrace{(\mu\hat{A}_{12} + B_1 G_f)}_{D} (L-L^*) = 0$$

Matrix D is $N_s \times N_f$. Therefore to guarantee $(L-L^*) \in \eta(D)$, D must be such that $N_f \geq N_s$ and is of full rank. With these conditions imposed

$$L = L^*. \quad (\eta \triangleq \text{Null Space})$$

Now, if it is also desired to place the slow eigenvalues to a desired accuracy, you would now iterate

$$M_{k+1} = \mu_M F_{11}^{-1} (\hat{F}_{12} L M_k + M_k F_{22} + M_k L \hat{F}_{12} - \hat{F}_{12}) \qquad M_o = -\mu_M F_{11}^{-1} \hat{F}_{12}.$$

Then select G_s to place N_s slow modes of $(A_s + B_s G_s)$ where

$$x_s(k+1) = \underbrace{[A_{11} - \mu A_{12} L]}_{A_s} x_s(k) + \underbrace{[(I-ML)B_1 - MB_2]}_{B_f} u(k). \tag{4.34}$$

Then,

$$\begin{aligned} u_s(k) &= G_s x_s(k) \\ &= G_s [(I-ML)x_1(k) - Mx_2(k)] \\ &= G_s (I-ML)x_1(k) - G_s M x_2(k). \end{aligned} \tag{4.35}$$

Our composite control is then

$$\begin{aligned} u_c(k) &= u_f(k) + u_s(k) \\ &= G_f L x_1(k) + G_f x_2(k) + G_s (I-ML)x_1(k) - G_s M x_2(k) \\ &= [G_f L + G_f (I-ML)] x_1(k) + [G_f - G_s M] x_2(k) \end{aligned} \tag{4.36}$$

and it places N_s slow λ's of $(A_s + B_s G_s)$ and N_f fast λ's of $(A_f + B_f G_f)$. The design has been carried out without ever handling a high order matrix.

Part II: Structure "H"

Given

$$\begin{bmatrix} x_1(k+1) \\ - - - \\ x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ - - - & - - - \\ \mu\hat{A}_{21} & \mu\hat{A}_{22} \end{array}\right] \begin{bmatrix} x_1(k) \\ - - - \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ - - \\ B_2 \end{bmatrix} u(k) \tag{4.37}$$

where i. $\underline{x}_1(k) \in R^{N_s}$, $\underline{x}_2(k) \in R^{N_f}$

ii. A_{ij}, B_j, $j = 1,2$ are real constant coefficient matrices

iii. μ is a small positive parameter such that

$$0 < \mu \leq \frac{1}{3\|A_{11}^{-1}\|\,(\|A_o\| + \|L_o\|\|\hat{A}_{21}\|)} \qquad A_o = \hat{A}_{22} - \hat{A}_{21}L_o.$$

Then we can iterate

$$L_{k+1} = A_{11}^{-1}(A_{12} + \mu L_k\hat{A}_{22} - \mu L_k\hat{A}_{21}L_k) \qquad L_o = A_{11}^{-1}A_{12} \tag{4.38}$$

to a desired accuracy.

We then form the slow subsystem

$$x_s(k+1) = A_s x_s(k) + B_s u_s(k)$$

where

$$A_s = A_{11} + \mu L\hat{A}_{21}$$

$$B_s = B_1 + LB_2.$$

We then design a slow feedback G_s to place the N_s slow eigenvalues of $(A_s + B_s G_s)$ at N_s slow locations.

Our partial feedback transformed system now looks like

$$\begin{bmatrix} x_s(k+1) \\ - - - \\ x_f(k+1) \end{bmatrix} = \left[\begin{array}{c|c} (A_{11} + \mu L\hat{A}_{21} + B_s G_s) & 0 \\ - - - - - - - - - & - - - - - \\ B_f G_s & \mu(A_{22} + \hat{A}_{21}L) \end{array}\right] \begin{bmatrix} x_s(k) \\ - - - \\ x_f(k) \end{bmatrix} + \begin{bmatrix} B_1 + LB_1 \\ - - - - - - - \\ -MB_1 + (I - ML)B_2 \end{bmatrix} u_f(k). \tag{4.39}$$

If this system were reblockdiagonalized through an M_k successive approximation iterative procedure (i.e. $x_f^*(k) = x_f(k) - M^* x_s(k)$), our designed slow

eigenvalues and original fast eigenvalues would be preserved. The only limitations on the accuracy would be how many iterations of (4.26) we used in establishing our reduced order subsystem.

When implemented on our original model

$$\begin{aligned} u_s(k) &= G_s x_s(k) \\ &= G_s[x_1(k) + L x_2(k)] \\ &= G_s x_1(k) + G_s L x_2(k) \end{aligned} \tag{4.40}$$

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11}+B_1G_s & A_{12}+B_1G_sL \\ \hline \mu\hat{A}_{21}+B_2G_s & \mu\hat{A}_{22}+B_2G_sL \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u_f(k). \tag{4.41}$$

Depending upon the shift of the annulus of slow eigenvalues in the z-plane due to the slow feedback,

$$\frac{\|\mu\hat{A}_{22} + B_2G_sL\|}{\|A_{11} + B_1G_s\|} = \mu^* \neq \mu \quad \text{in general.}$$

Thus, our new system will have the form

$$\begin{bmatrix} x_1(k+1) \\ \hline x_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline \mu^*F_{21} & \mu^*F_{22} \end{array}\right] \begin{bmatrix} x_1(k) \\ \hline x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u(k). \tag{4.42}$$

At this stage, the slow eigenvalues have been placed at N_s slow locations to a desired accuracy using a reduced order slow subsystem (N_s). Plus, the fast eigenvalues have been untouched.

If you also want to move the fast modes without effecting the slow modes, we can prove a dual invariance property of L if the system is of the proper form. To block diagonalize system (4.42) two matrix iterations must converge and thus our μ bounds should be retested. These matrix iterations are

$$L^*_{k+1} = F_{11}^{-1}[F_{12} + L^*_k F_{22} - L^*_k F_{21} L^*_k] \tag{4.43}$$

$$M_{k+1} = [F_{21} + F_{22}M_k + F_{21}L^*M_k + M_k L^* F_{21}]F_{11}^{-1}. \tag{4.44}$$

Lemma 10: If $N_s \geq N_f$ and $A_{21} + B_2 G_s$ is of full rank, then L is invariant to the partial feedback

$$u_s(k) = G_s x_1(k) + G_s L x_2(k).$$

Proof: In this case, the fast subsystem does not change, therefore

$$F_{22} - F_{21}L^* = A_{22} - A_{21}L$$

$$A_{22} + B_2 G_s L - (A_{21} + B_2 G_s)L^* = A_{22} - A_{21}L$$

$$B_2 G_s[L-L^*] + A_{21}[L-L^*] = 0$$

$$\underbrace{[A_{21} + B_2 G_s]}_{D_{(N_f \times N_s)}}[L-L^*] = 0.$$

Therefore, if $N_s \geq N_f$, and D is of full rank, such that

$$[L-L^*] \notin \eta[D],$$

then $L = L^*$. If this condition holds, we would then iterate

$$M_{k+1} = \mu[\hat{F}_{21} + \hat{F}_{22}M_k + \hat{F}_{21}LM_k + M_k L\hat{F}_{21}]F_{11}^{-1}. \tag{4.45}$$

Then, using either (4.37) and (4.45) or (4.43) and (4.44) we would place N_f fast eigenvalues using the fast subsystem

$$x_f(k+1) = A_f x_s(k) + B_f u_f(k)$$

where

$$A_f = \mu\hat{A}_{22} - \mu\hat{A}_{21}L_k$$

$$B_f = -MB_1 + (I + ML)B_2$$

we then find a fast control

$$u_f(k) = G_f x_f(k)$$

$$= G_f[-Mx_1(k) + (I+ML)x_2(k)]$$

$$= -G_fMx_1(k) + G_f(I+ML)x_2(k) \qquad (4.46)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11}+B_1G_s-B_1G_fM, & A_{12}+B_1G_sL^*+B_1G_f(I+ML) \\ \mu(\hat{A}_{21})+B_2G_s-B_2G_fM, & \mu(\hat{A}_{22})+B_2G_sL^*+B_2G_f(I+ML) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}. \qquad (4.47)$$

In the next section, we will give an example based on model "H." In the following section we will propose an algorithm for the general model.

4.6. Example – Structure "H"

Given

$$\underline{x}(k+1) = \left[\begin{array}{cc|cc} 1 & .6 & .4 & .8 \\ -.2 & 1.3 & 0 & .7 \\ \hline 0 & .1 & .05 & .7 \\ -.1 & .1 & 0 & .1 \end{array}\right] \underline{x}(k) + \left[\begin{array}{c} 1 \\ 1 \\ \hline 1 \\ 1 \end{array}\right] u(k).$$

The open loop eigenvalues are at

$$\lambda_{1,2} = 1.16 \pm .37j$$

$$\lambda_3 = .1$$

$$\lambda_4 = .0235.$$

Therefore $N_s = N_f = 2$

The desired closed-loop eigenvalues are at

$$\lambda_1 = .98$$

$$\lambda_2 = .96$$

$$\lambda_3 = .2$$

$$\lambda_4 = .15$$

Iterating the L_k matrix to 6 iterations, we get

$$L = \left[\begin{array}{c|c} .38416 & .45947 \\ \hline .077 & .651707 \end{array}\right]$$

and a slow subsystem of

$$x_s(k+1) = \left[\begin{array}{c|c} 1.045947 & .515637 \\ \hline -.134829 & 1.227129 \end{array}\right] x_s(k) + \left[\begin{array}{c} 1.8436 \\ \hline 1.7287 \end{array}\right] u_s(k).$$

Using LINSYS [16] to place our slow modes, we found

$$G_s = [.077118 \vdots -.30607]$$

$$u_s(k) = [.077118, -.30607, .0060567, -.16403455]x(k)$$

and our partially closed-loop system looks like

$$x(k+1) = \begin{bmatrix} 1.0771 & .29393 & .40606 & .63597 \\ -.12288 & .99393 & .0060567 & .53597 \\ .077118 & -.20607 & .056057 & -.064035 \\ -.022882 & -.20607 & .0060567 & -.064035 \end{bmatrix} x(k)$$

with eigenvalues

$$\lambda_1 = .98005$$

$$\lambda_2 = .95994$$

$$\lambda_3 = .099484$$

$$\lambda_4 = .023592.$$

We now reiterate both the L_k and M_k matrix recursions to show the invariance property of L_k since $(A_{21} + B_2 G_s)$ is obviously nonsingular.

$$L^* = \begin{bmatrix} .38417 & .45948 \\ .077 & .651713 \end{bmatrix}$$

$$M = \begin{bmatrix} .0605 & -.19712 \\ -.031422 & -.138776 \end{bmatrix}$$

Our fast subsystem is defined by

$$x_f(k+1) = \begin{bmatrix} .069815 & -.162899 \\ -.018603 & -.208847 \end{bmatrix} x_f(k) + \begin{bmatrix} 1.4064 \\ 1.5752 \end{bmatrix} u_f(k).$$

Again, placing our desired fast poles

$$G_f = [.64234,\ -.42944]$$

$$u_f(k) = [-.048113,\ .059422,\ .62834,\ -.41281]\underline{x}(k)$$

which gives closed loop eigenvalues of

$$\lambda_1 = .97999,\ \lambda_2 = .96001,\ \lambda_3 = .19972,\ \lambda_4 = .15028.$$

4.7. Independent Slow-Fast Modeling for Reduced-Order Eigenvalue Placement

In the previous two sections, we outlined high-accuracy reduced-order modeling algorithms for the purpose of eigenvalue placement in systems of structures "H" and "V." While we were able to prove some additional properties of certain system matrices which could save computations, both algorithms suffered a common problem. Given a system fitting model "V," we found that, in general, we had to run two iterative matrix recursion equations to place the N_s slow modes opposed to one matrix recursion to place the N_f fast modes. The dual problem exists for model "H." This could be a severe time setback in a multiprocessing control environment. What we propose now is to apply Section 2.5 of Chapter 2 and develop an algorithm similar to the previous section such that it can be applied to any system of our general form ($0 \leq j \leq 1$). Also, the dependent $Q_k(M_k)$ matrix recursions will never be needed.

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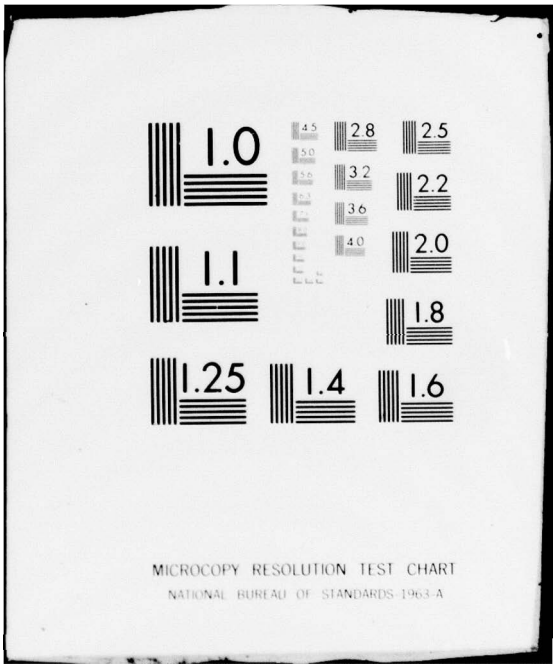
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We now state a lemma which establishes a logical sufficient condition to block diagonalize our general model using either transformation.

Lemma 11: Given a linear shift-invariant discrete-time system that fits model (2.1), to block diagonalize this system by either transformation "F" or "S," it is sufficient that

$$\mu \leq \min(\mu_{f_{max}}, \mu_{s_{max}}). \tag{4.48}$$

Proof: From Section 2.5 of Chapter 2, for transformation "F" to converge

$$0 \leq \mu_f < \frac{1}{3\, A_{11}^{-1}\,(\,\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}A_{12} + \hat{A}_{21}A_{11}^{-1}\,\hat{A}_{12}\,)} = \mu_{f_{max}}. \tag{4.49}$$

Likewise for transformation "S" to converge

$$0 \leq \mu_s < \frac{1}{3\, A_{11}^{-1}(\,\hat{A}_{22}-\hat{A}_{21}A_{11}^{-1}\hat{A}_{12} + \hat{A}_{21}\,A_{11}^{-1}\hat{A}_{12}\,)} = \mu_{s_{max}}. \tag{4.50}$$

Thus the logical conclusion since each bound on μ (4.49), (4.50) was true for any $j \in [0,1]$. Now, for any μ satisfying Lemma 11, we propose the following straightforward procedure. Given

$$\begin{bmatrix} \underline{x}_1(k+1) \\ \hline \underline{x}_2(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & \mu^{(1-j)}\hat{A}_{12} \\ \hline \mu^{j}\hat{A}_{21} & \mu\hat{A}_{22} \end{array}\right] \begin{bmatrix} \underline{x}_1(k) \\ \hline \underline{x}_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} u(k) \tag{4.51}$$

where

i. $\underline{x}_1(k) \in R^{N_s}$, $x_2(k) \in R^{N_f}$, $u(k) \in R^M$

ii. (A,B) is a strongly controllable pair in μ^k

iii. A_{ij}, B_j are real constant coefficient matrices

iv. $0 \leq j \leq 1$

v. μ satisfies Lemma 11

vi. $A_{12} = \mu^{(1-j)}\hat{A}_{12}$, $A_{22} = \mu\hat{A}_{22}$, $A_{21} = \mu^{j}\hat{A}_{21}$.

Then define a slow subsystem

$$A_s = A_{11} + P_k^s A_{21}$$

$$P_{k+1}^s = A_{11}^{-1}(A_{12} + P_k^s A_{22} - P_k^s A_{21} P_k^s) \qquad P_o^s = A_{11}^{-1} A_{12}$$

with control matrix

$$B_s = B_1 + P_k^s B_2$$

such that

$$x_s(k+1) = A_s x_s(k) + B_s u_s(k) \tag{4.52}$$

and define a fast subsystem,

$$A_f = A_{22} + A_{12} P_k^f$$

$$P_{k+1}^f = (A_{22} P_k^f + P_k^f A_{12} P_k^f - A_{21}) A_{11}^{-1} \qquad P_o = -A_{21} A_{11}^{-1}$$

with control matrix

$$B_f = P_k^f B_1 + B_2$$

such that

$$x_f(k+1) = A_f x_f(k) + B_f u_f(k).$$

The strong controllability condition in μ^k on our original system guarantees controllability of the pairs

$$(A_s, B_s), \quad (A_f, B_f).$$

The designer now has the option as to which subsystem he wants to work with. Let us say, for example, that the designer first wishes to optimize or place the slow modes. Then, we find a <u>slow</u> feedback

$$u_s(k) = G_s x_s(k)$$

for (4.52), where G_s is the product of an N_s order pole-placement procedure or a slow quadratic performance criteria

$$J_s(x_s, u_s) = \tfrac{1}{2} \sum_{k=0}^{\infty} (x_s'(k) Q_s x_s(k) + u_s'(k) R u_s(k))$$

where (A,D_s) is an observable pair, $D_s'D_s = Q_s$

$$R \geq 0 \quad Q_s \geq 0$$

$$u_s(k) = G_s x_s(k)$$
$$= -(R + B_s'K\,B_s)^{-1}(B_s'KA_s)x_s(k)$$

where K is the solution to

$$K = A_s'KA_s + Q_s - (A_s'KB_s)(R + B_s'KB_s)^{-1}(B_s'KA_s).$$

Either method of finding G_s is acceptable as long as $u_s(k) = G_s x_s(k)$ places the N_s slow modes at N_s slow locations (i.e. the slow modes remain in some stable annulus sufficiently spread from the fast modes which are within some radius $R(\mu)$ of the origin in the z-plane). This requirement is necessary if we are to next carry out a reduced order fast design.

Once G_s has been obtained, to implement this control on the original system,

$$u_s(k) = G_s x_s(k)$$
$$= G_s[x_1(k) + P_s x_2(k)]$$
$$= [G_s \vdots G_s P_s]\underline{x}(k). \tag{4.53}$$

This simple form of the solution enables our control to span only the slow controllable subspace, and thus keep all fast modes invariant to $0(\mu^k)$. This is seen much in the same way as in the previous chapter. Our "S" transformation, when applied to our general model, results in

$$\begin{bmatrix} x_s(k+1) \\ \hline x_f(k+1) \end{bmatrix} = \left[\begin{array}{c|c} A_s & 0 \\ \hline 0 & A_f \end{array}\right] \begin{bmatrix} x_s(k) \\ \hline x_f(k) \end{bmatrix} + \begin{bmatrix} B_s \\ \hline B_f \end{bmatrix} u(k)$$

where
$$A_s = A_{11} + P_k^s A_{21}$$
$$B_s = B_1 + P_k^s B_2$$
$$A_f = A_{22} - A_{21} P_k^s$$
$$B_f = -Q_k^s B_1 + (I - Q_k^s P_k^s) B_2$$

with the feedback $u_s(k) = G_s x_s(k)$ applied

$$\begin{bmatrix} x_s(k+1) \\ \hline x_f(k+1) \end{bmatrix} = \left[\begin{array}{c|c} (A_s + B_s G_s) & 0 \\ \hline B_f G_s & A_f \end{array}\right] \begin{bmatrix} x_s(k) \\ \hline x_f(k) \end{bmatrix} + \begin{bmatrix} B_s \\ \hline B_f \end{bmatrix} u(k). \tag{4.54}$$

We could block diagonalize this system by letting $(u(k) = 0)$

$$x_f^*(k) = x_f(k) - T x_s(k)$$
$$x_f^*(k+1) = x_f(k+1) - T x_s(k+1)$$
$$= [B_f G_s - T(A_s + B_s G_s) + A_f T] x_s(k) + A_f x_f^*(k)$$

we want T such that

$$B_f G_s - T(A_s + B_s G_s) + A_f T = 0. \tag{4.55}$$

However, since $(A_s + B_s G_s)$ and A_f have no eigenvalues in common, the existence of a solution to this Lyapunov type equation is guaranteed by [14]. Thus, the eigenvalues of (4.54) will be determined by the eigenvalues of $(A_s + B_s G_s)$ and A_f preserving the accuracy of our designed slow eigenvalues and the original fast eigenvalues to $0(\mu^k)$.

If the designer now wants to generate a reduced order fast subsystem for design purposes, he must first update the system matrices

$$A_{11}^* = A_{11} + B_1 G_s, \quad A_{12}^* = A_{12} + B_1 G_s P^s$$
$$A_{21}^* = A_{21} + B_2 G_s, \quad A_{22}^* = A_{22} + B_2 G_s P^s \tag{4.56}$$

Since $\|P_k^s\| \triangleq 0(\mu^{1-j})$, if $\|B_1 G_s\| \triangleq 0(1)$ and $\|B_2 G_s\| \triangleq 0(\mu^j)$, or less, our bounds on μ will be unchanged. The bound on $\|B_1 G_s\|$ is easily satisfied if we remember the set of eigenvalues we are considering are within some radius $1+\epsilon$ of the origin of the z-plane. The bound on $\|B_2 G_s\|$ is somewhat limiting and suggests the general form $B_2 = \mu^j \hat{B}_2$. However, properties like this come down to a case by case study. In general, after the updates (4.56), define the fast subsystem

$$A_f = A_{22}^* + A_{12}^* P_k^f$$

$$P_{L+1}^f = (A_{22}^* P_L^f + P_L^f A_{12}^* P_L^f - A_{21}^*) A_{11}^{*-1} \qquad P_o = -A_{21}^* A_{11}^{*-1}$$

with control matrix

$$B_f = P_L^f B_1 + B_2$$

such that

$$x_f(k+1) = A_f x_f(k) + B_f u_f(k). \tag{4.57}$$

Find a fast feedback matrix G_f through pole placement or a fast N_f order optimal linear regulator solution. Either way, we have N_f fast modes at N_f new <u>fast</u> locations. To implement on actual system,

$$\begin{aligned} u_f(k) &= G_f x_f(k) \\ &= G_f[P_L^f x_1(k) + x_2(k)] \\ &= [G_f P_L^f \vdots G_f] x(k). \end{aligned}$$

Again, this simple form of the solution enables our control to span only the fast controllable subspace, and thus keep all slow modes invariant to $0(\mu^L)$.

Combining this with our previously defined slow control we get

$$\begin{aligned} u(k) &= u_s(k) + u_f(k) \\ &= [G_s + G_f P_1^f \vdots G_f + G_s P_k^s] x(k) \end{aligned} \tag{4.58}$$

which places N_s slow poles at

$$\lambda(A_s + B_s G_s) + 0(\mu^k)$$

and N_f fast poles at

$$\lambda(A_f + B_f G_f) + 0(\mu^L).$$

The order of fast and slow designs is not important as long as the dual procedure is carried out. Again note that we could have specified our control matrices to be of some special form (i.e. $\mu^j \hat{B}_2$, $\mu^{(1-j)} \hat{B}_1$) and perhaps simplify the computations. Instead, by imposing the less conservative form, $\|B_i\| \triangleq 0(1)$ or less, we encompass a greater class of systems possessing open-loop two-time-scale properties.

In the next section, we give an 8th order power system example. A flow chart illustrating this design procedure is given on the next page.

4.8. Example - 8th Order Power System

In [19], the model of an isolated mixed power system is given as,

$$\dot{\underline{x}} = \begin{bmatrix} -.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4.75 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16667 & 16667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & -.08 & -.07467 & -.112 & 3.9944 & 10 & -.92778 & -9.10111 \\ 0 & 0 & 0 & 0 & .20 & .50 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.31944 & 0 & -1.38889 & .27778 \\ 0 & .01 & .00933 & .014 & -.06319 & 0 & .11597 & .11236 \end{bmatrix} \underline{x} + \begin{bmatrix} 4 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 10 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \underline{u}$$

with open loop eigenvalues of

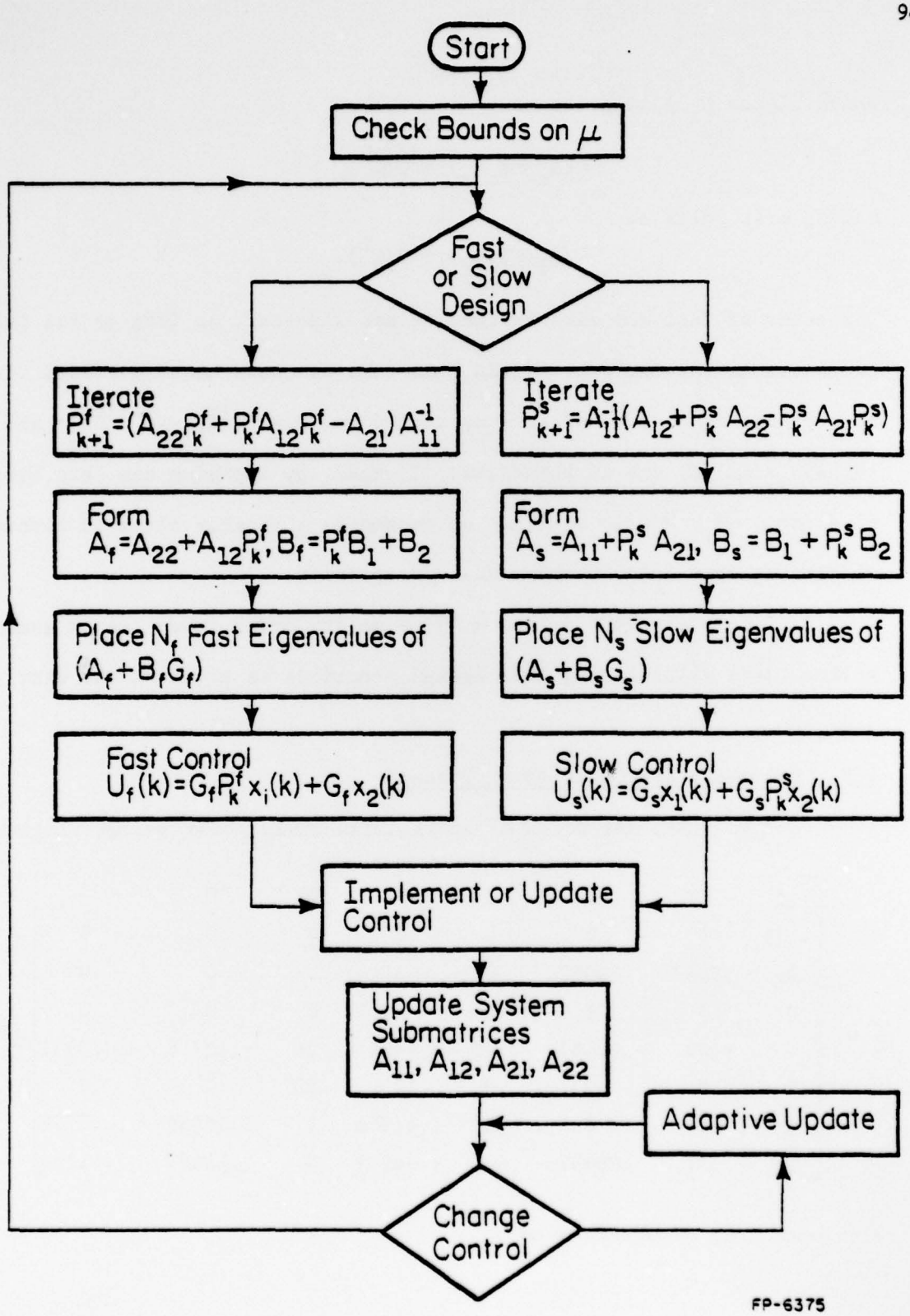

Figure 4.1. Flowchart for independent slow/fast subsystem design.

$$\lambda_1 = -5.0066 \qquad \lambda_5 = -.086912 + .33047j$$

$$\lambda_2 = -4.3542 \qquad \lambda_6 = -.086912 - .33047j$$

$$\lambda_3 = -1.9939 \qquad \lambda_7 = -.37062$$

$$\lambda_4 = -1.3884 \qquad \lambda_8 = -.075716$$

which would suggest $N_s = N_f = 4$.

This system is then run through a continuous singularly perturbed modeling software package in an attempt to put this system into continuous two-time-scale form. As a result, the above system is transformed by the permutation matrix

$$P = [e_1, e_3, e_6, e_8, e_5, e_4, e_7, e_2].$$

This system is to be controlled in a sampled-data environment where $T = .9$ sec. The control design is then best carried out in the z-plane. The corresponding discrete model is given as

$$\underline{x}(k+1) = \begin{bmatrix} .83527 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .096874 & .86071 & 0 & 0 & 0 & 0 & 0 & .0293 \\ -.00223 & -.005 & .88241 & -.25361 & .040567 & -.0025737 & -.0251 & -.00111 \\ .0068261 & .014284 & -.029647 & .92793 & 0 & .00576 & .059 & .00227 \\ -.029713 & -.0609 & 2.0283 & -2.303 & .08842 & -.021168 & -.22386 & -.008 \\ .047959 & .75863 & 0 & 00 & 0 & .1653 & 0 & .022615 \\ -.012274 & -.02676 & 1.2091 & -1.4005 & .16097 & -.01326 & .15593 & -.00558 \\ .81558 & 0 & 0 & 0 & 0 & 0 & 0 & .01111 \end{bmatrix} \underline{x}(k)$$

$$+ \begin{bmatrix} 3.2946 & 00 \\ .15247 & 00 \\ -.0025695 & .2936 \\ .010316 & -.038309 \\ -.050766 & 2.7624 \\ .056549 & 00 \\ -.01494 & 1.4729 \\ 2.4774 & 0 \end{bmatrix} \underline{u}(k) \qquad (4.59)$$

with system eigenvalues

$$\lambda_1 = .874 + .1692j \qquad \lambda_5 = .2866$$
$$\lambda_2 = .874 - .1692j \qquad \lambda_6 = .1653$$
$$\lambda_3 = .8607 \qquad \lambda_7 = .02$$
$$\lambda_4 = .8353 \qquad \lambda_8 = .0111.$$

Thus, our two-time-scale eigenvalue separation is preserved.

After testing (4.59) with some design software [Appendix B], we found (4.59) to be of general form with $\mu = .259904$. If a reduced order slow model is desired, transformation "S" is generated using 2 iterations of P_k^s giving

$$P_k^s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .0344832 \\ .048604 & -.0011815 & -.0075897 & -.0003338 \\ .0118520 & .0072417 & .0877587 & .0028456 \end{bmatrix}.$$

Since $0(\mu^3)$ accuracy will be sufficient for slow modes. We thus generate the fourth order model

$$x_s(k+1) = \begin{bmatrix} .83527 & 0 & 0 & 0 \\ .125 & .8607 & 0 & 0 \\ -.0039 & -.008567 & .971817 & -.3549061 \\ .008065 & .01671 & .1005 & .777731 \end{bmatrix} x_s(k)$$
$$+ \begin{bmatrix} 3.2946 & 0 \\ .2379 & 0 \\ -.00582 & .416685 \\ .01586 & .123691 \end{bmatrix} \underline{u}_s(k) \qquad (4.60)$$

with eigenvalues

$$\lambda_1 = .87477 + .16202j$$

$$\lambda_2 = .87477 - .16202j$$

$$\lambda_3 = .80071$$

$$\lambda_4 = .83527.$$

The designer now wishes to move the open loop slow modes to $(.9 \pm .05j, .85, .80)$. Using LINSYS [16] on (4.60), we obtain a feedback gain

$$G_s = \begin{bmatrix} .075043 & -.60196 & 0 & 0 \\ -.073167 & -.036092 & -.46533 & .76279 \end{bmatrix}$$

and when implemented on our actual system this becomes

$$\begin{aligned} u_s(k) &= G_s x_s(k) \\ &= G_s[x_1(k) + P_1^s x_2(k)] \\ &= [G_s \vdots G_s P_1^s]\underline{x} \\ &= \left[\begin{array}{cccc:cccc} .075043 & 0.60196 & 0 & 0 \\ -.073167 & -.036092 & -.46533 & .76279 \\ \hdashline 0 & 0 & 0 & -.020758 \\ -.013577 & .0060737 & .070473 & .0010813 \end{array}\right]\underline{x} \end{aligned}$$

and gives closed loop eigenvalues of

$$\lambda_1 = .9 - .05j \qquad \lambda_5 = .28400$$

$$\lambda_2 = .9 + .05j \qquad \lambda_6 = .1653$$

$$\lambda_3 = .80976 \qquad \lambda_7 = .019959$$

$$\lambda_4 = .84142 \qquad \lambda_8 = .011109$$

which verifies the $0(\mu^3)$ accuracy.. Now, if the designer also chooses to move the fast modes, the system submatrices are updated and the "F"

transformation is updated using 3 iterations of P_k^f giving

$$P_3^f = \begin{bmatrix} .143835 & .5253375 & -.988291 & .181422 \\ .066625 & -.9768864 & 0 & 0 \\ .059635 & .45438 & -1.3080671 & .4155951 \\ -.896520 & -.378294 & 0 & 0 \end{bmatrix}$$

and a fourth order fast subsystem model

$$\underline{x}_f(k+1) = \begin{bmatrix} .0148718 & -.00261 & -.0146517 & -.00164 \\ 0 & .1653 & 0 & -.00864 \\ .0933589 & -.001 & .28884 & .004763 \\ 0 & 0 & 0 & .01111 \end{bmatrix} x_f(k)$$

$$+ \begin{bmatrix} .50762 & 2.4653 \\ .1271 & 0 \\ .2584564 & 1.0729305 \\ -.533954 & 0 \end{bmatrix} u_f(k) \qquad (4.61)$$

with eigenvalues at

$$\lambda_1 = .01996$$

$$\lambda_2 = .28375$$

$$\lambda_3 = .16530$$

$$\lambda_4 = .01111.$$

The designer wishes to place these eigenvalues at (.2, .1, .05, .01). Using LINSYS [16] on (4.61) we obtain the feedback

$$G_f = \begin{bmatrix} 0 & .09359 & 0 & -.20918 \\ -.044114 & .16405 & -.12578 & -.011122 \end{bmatrix}.$$

When implemented on actual system becomes

$$
\begin{aligned}
u_f(k) &= G_f x_f(k) \\
&= G_f[P_f x_1(k) + x_2(k)] \\
&= [G_f P_f \vdots G_f]\underline{x}(k) \\
&= \left[\begin{array}{cccc:cccc}
.19377 & -.01229 & 0 & 0 & 0 & .09359 & 0 & -.20918 \\
.0070544 & -.23638 & .20813 & -.060278 & -.044114 & .16405 & -.12578 & -.011122
\end{array}\right]\underline{x}(k).
\end{aligned}
$$

Thus, the net control

$$u(k) = u_s(k) + u_f(k)$$

results in closed loop eigenvalues

$$
\begin{aligned}
\lambda_1 &= .89999 + .050005j & \lambda_5 &= .20000 \\
\lambda_2 &= .89999 - .050005j & \lambda_6 &= .099952 \\
\lambda_3 &= .84268 & \lambda_7 &= .056108 \\
\lambda_4 &= .80758 & \lambda_8 &= .01101.
\end{aligned}
$$

5. CONCLUSION

In this thesis the concept of a two-time-scale linear shift-invariant discrete-time system is introduced. Properties of such systems have been well documented for the continuous case. These properties include reduced order modeling, separate slow-fast control law design, boundry layer phenomena, etc. It was the purpose of this thesis to develop a model for discrete-time systems which possessed as many of these properties as possible without restricting the model to isolated special cases.

The main results will now be reviewed very briefly. First, criteria for the two-time-scale property of a system using matrix norm conditions are proposed. Then a model satisfying these conditions is introduced. If a two-time-scale system matrix does not fit the model, a simple procedure of permutation and scaling is outlined such that the transformed system matrix will fit the model. A pair of dual transformations are proposed which transform our general model into slow-fast block diagonal form. Conditions are given for the existence of both transformations. The difference between a degenerate and normal system is briefly discussed.

The recursive property of discrete-time systems is used to reveal the two-time-scale nature of our model by iteratively solving for slow or "steady-state" models. We thus introduce reduced order modeling and the concept of a "boundry layer" for our class of systems.

Separate slow and fast control law design is then established for the purpose of eigenvalue placement. Controllability conditions are discussed and an algorithm is proposed for reduced-order control law design which maintains is high level of accuracy when implemented on the high

order system. Also, the basic differences between discrete two-time-scale systems and continuous singularly perturbed systems are discussed.

APPENDIX A

PROOF OF CONDITIONS NEEDED FOR BLOCK DIAGONALIZATION

The proof given in [6] is applied several times in this thesis. For the convenience of the reader it is restated here in context.

It is required that L exists such that

$$A_{22}L - LA_{11} + LA_{12}L - A_{21} = 0 \tag{A1}$$

Let $L_0 = A_{22}^{-1}A_{21}$ $\quad A_0 = A_{11} - A_{12}L_0$

$B_1 = A_{11} - A_{12}L$ $\quad B_2 = A_{22} + LA_{12}$

seek L of the form $L = L_0 + D$. Where D is a real root of

$$DA_0 - (A_{22} + L_0A_{12})D - DA_{12}D + L_0A_0 = 0. \tag{A2}$$

The following lemma gives a sufficient condition for the existence and uniqueness of a real root D and establishes a bound for its norm $\|D\|$. It also formulates a convergent procedure for iterative calculation of D.

Lemma 1: If A_{22} is nonsingular and if

$$\|A_{22}^{-1}\| < \tfrac{1}{3}\|A_0\| + \|A_{12}\|\|L_0\|)^{-1} \tag{A3}$$

then a unique real root of (A2) exists satisfying

$$0 < \|D\| < \frac{2\|A_0\|\|L_0\|}{\|A_0\| + \|A_{12}\|\|L_0\|}. \tag{A4}$$

This root is an asymptotically stable equilibrium of the difference equation

$$D_{k+1} = A_{22}^{-1}(L_0A_0 + D_kA_0 - L_0A_{12}D_k - D_kA_{12}D_k) \equiv f(D_k) \tag{A5}$$

and its domain of attraction encompasses the set of matrices defined by (A4).

Proof: For $a = \|A_0\|$, $b = \|A_{12}\|\|L_0\|$, $c = \|A_{22}^{-1}\|$ and

$$d_k = \frac{\|D_k\|}{\|A_0\|\|L_0\|} \tag{A6}$$

we obtain from (A5)

$$d_{k+1} < c[1 + (a+b)d_k + abd_k^2] < c\left(\frac{a+b}{2}\, d_k + 1\right)^2 \tag{A7}$$

and we analyze the upper bound of d_k defined by

$$\bar{d}_{k+1} = c\left(\frac{a+b}{2}\, \bar{d}_k + 1\right)^2 . \tag{A8}$$

Obviously $d_k < \bar{d}_k$ for all k. When $c < \frac{1}{2}(a+b)^{-1}$ then the scalar difference equation (A8) has two real equilibrium points d' and $d'' > d'$ and d' lies in the interval $[0, 2(a+b)^{-1}]$. For all $\bar{d}_k \neq d'$ in this interval we have $|\bar{d}_{k+1} - d'| < |\bar{d}_k - d'|$, which proves that D_k is bounded, that is

$$0 < \|d_k\| < 2(a+b)^{-1} \tag{A9}$$

holds for all $k = 1,2,\ldots$, if it holds for $k = 0$. Substituting $\delta D_k = D_k - D$ into (A5) we get

$$\delta D_{k+1} = A_{22}^{-1}[\delta D_k(A_0 - A_{12}D) - (L_0 + D)A_{12}\delta D_k - \delta D_k A_{12}\delta D_k] \tag{A10}$$

which, using $\|\delta D_k\| \equiv \nu_k < \|D_k\| + \|D\|$, implies

$$\nu_{k+1} < c[a + b + 3\|A_{12}\|\|D\| + \|A_{12}\|\|D_k\|]\nu_k . \tag{A11}$$

When both $\|D_k\|$ and $\|D\|$ satisfy (A9), then (A11) yields

$$\nu_{k+1} < c\left(a+b+\frac{8ab}{a+b}\right)\nu_k < 3c(a+b)\nu_k \tag{A12}$$

and, hence, if $c < \frac{1}{3}(a+b)^{-1}$, as required by (A3), then $f(D_k)$ in (A5) is a contraction mapping and D is its fixed point. We complete the proof by noting that ν_k is a Lyapunov function and $\nu_{k+1} - \nu_k < 0$ for all D_k satisfying (A9).

Using (A5) with the initial condition $D_0 = 0$ we can calculate D iteratively. By (A3) and (A12) after k iterations the relative error is

$$\frac{\|D_k - D\|}{\|D\|} < [3\|A_{22}^{-1}\|(\|A_0\| + \|A_{12}\|\|L_0\|)]^k \tag{A13}$$

and it decreases as $\|A_{22}^{-1}\|$ and $\|A_0\|$ decrease, that is as the ill-conditioning of the system increases. When we need M to be a real root of

$$B_1M - MB_2 + A_{12} = 0. \tag{A14}$$

The following lemma formulates a convergent iterative method for solving (A14).

Lemma 2: Under the conditions of Lemma 1 the solution M of (A14) is the asymptotically stable equilibrium of the linear difference equation

$$M_{k+1} = [(A_{11} - A_{12}L)M_k - M_kLA_{12}]A_{22}^{-1} + A_{12}A_{22}^{-1}. \tag{A15}$$

Proof: For $m_k = \|M_k - M\|$ we obtain from (A14) and (A15)

$$m_{k+1} < c[a+b+2\|A_{12}\|\|D\|]m_k \tag{A16}$$

and, by virtue of (A4)

$$m_{k+1} < c\left[a+b+\frac{4ab}{a+b}\right]m_k. \tag{A17}$$

Thus, $m_{k+1} < m_k$ if $c < \frac{1}{2}(a+b)^{-1}$ which is satisfied by (A3).

APPENDIX B

COMPUTER PACKAGES

Through the use of the IMSL matrix analysis computer library and the AG210 computer graphics library, an extensive discrete two-time-scale systems software packages has been compiled. This software accepts an arbitrary linear shift-invariant discrete-time system and determines whether or not it possesses a two-time-scale property, what order reduction is possible, μ, comparitive time responses, reduced order eigenvalue placement, etc. Rather than list the source program here, the author welcomes any party that may be interested to contact him.

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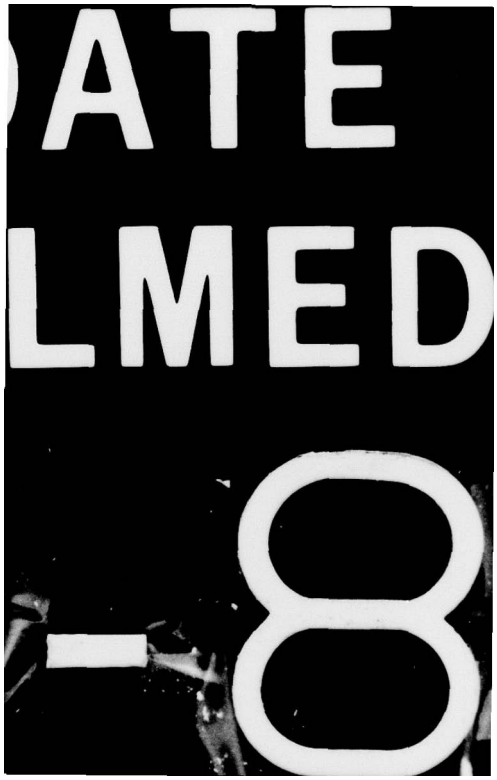
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